

# INTEGRABLE AND PROPER ACTIONS ON $C^*$ -ALGEBRAS, AND SQUARE-INTEGRABLE REPRESENTATIONS OF GROUPS

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**ABSTRACT.** We propose a definition of what should be meant by a *proper* action of a locally compact group on a  $C^*$ -algebra. We show that when the  $C^*$ -algebra is commutative this definition exactly captures the usual notion of a proper action on a locally compact space. We then discuss how one might define a *generalized fixed-point algebra*. The goal is to show that the generalized fixed-point algebra is strongly Morita equivalent to an ideal in the crossed product algebra, as happens in the commutative case. We show that one candidate gives the desired algebra when the  $C^*$ -algebra is commutative. But very recently Exel has shown that this candidate is too big in general. Finally, we consider in detail the application of these ideas to actions of a locally compact group on the algebra of compact operators (necessarily coming from unitary representations), and show that this gives an attractive view of the subject of square-integrable representations.

There is a variety of situations in which actions of locally compact groups on non-commutative  $C^*$ -algebras appear to be “proper” in a way analogous to proper actions of groups on spaces. See for example [OP1, OP2, Ks, Rf7, Ma, Qg, Rf8, QR, Ab, E1, GHT]. We propose here a simple definition which seems to capture this idea reasonably well. We indicate a variety of examples, but we only explore two basic ones in detail. Namely, we show that when the  $C^*$ -algebra is commutative our definition does capture exactly the usual definition of a proper action on a space. Then we show that when the  $C^*$ -algebra is the algebra of compact operators on a Hilbert space our definition is very closely related to square-integrable representations (not necessarily irreducible) of groups, and gives an attractive view-point on this venerable subject.

Our definition of proper actions is closely related to ideas of “integrable” actions which occur in various places, especially in the literature concerning actions on von Neumann algebras [CT,Pa,S]. We give here a definition of “integrable” actions for  $C^*$ -algebras which appears to be the right analogue of that for von Neumann algebras. We see that every proper action is integrable, but not conversely. But it is useful to see that some of the basic properties of proper actions come just from the fact that they are integrable.

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I had earlier given a tentative definition of proper actions [Rf7], which was adequate to treat some interesting examples. But that definition assumed the existence of a dense subalgebra with certain properties, and so was not intrinsic. The definition proposed here is intrinsic, and includes my older definition. But the definition given here must still be viewed as tentative, since I have not yet been able to relate it strongly to the crossed product  $C^*$ -algebra for the action in the way done in [Rf7] (and there are many other aspects which also still need to be explored).

The main goal is to define a suitable generalized fixed-point algebra (corresponding to the orbit space of a proper action on a space), which will in a natural way be strongly Morita equivalent to an ideal in the crossed product algebra, as done in [Rf7]. In an earlier version of this paper I had proposed a candidate for this generalized fixed-point algebra, and shown that it works correctly when the algebra acted upon is Abelian (Theorem 6.5 below). But in a very recent preprint [E2] Exel gives a natural example showing that in general the candidate which I had proposed is too big. He also gives a penetrating analysis of the difficulties involved, already for the case when the group which acts is Abelian. He does this within the context of making strong progress on his project of determining which actions on  $C^*$ -algebras are dual actions on Fell bundles. But this leaves unresolved the question of whether there is an intrinsic definition of the generalized fixed-point algebra.

In spite of this unsatisfactory situation, it seems to me worthwhile publishing what I discovered. There is not much overlap between the very recent paper of Exel and the present paper, and Exel makes use of some of the examples and results of the present paper. Also, I needed to sort out some of the issues discussed here for use in connection with several of my projects concerning quantization. (And even the classical notion of proper actions on spaces is one of continuing strong interest [BCH, GHT].)

Actually, the definition of proper actions which we give here was strongly stimulated by a slightly earlier paper of Exel [E1] (which in turn built on [Rf7]). In fact our definition almost appears explicitly in [E1]. The main difference is that here we emphasize the order properties of  $C^*$ -algebras while in [E1] the emphasis is on unconditional integrability (as it is in [E2] also). I am very grateful to Exel for quite helpful comments about all these matters.

We will see that our definition of proper actions is closely related to earlier notions of integrable elements discussed in [Ld, OP1, OP2, Qg, QR]. It is also closely related to the notion of  $C^*$ -valued weights on  $C^*$ -algebras which was introduced recently by Kustermans [Ku], for fairly different reasons involving Haar measures for quantum groups. (I thank Kustermans for some helpful comments on this matter.) One can in turn ask what should be meant by proper actions of quantum groups. Integrable actions of a fairly wide class of quantum groups (namely Kac algebras) acting on von Neumann algebras are discussed in section 18.19 of [S]. The action of any compact quantum group on a  $C^*$ -algebra should be proper, and indeed in this case one obtains the kind of relations between the fixed point algebra and the crossed product algebra which one expects [Ng]. One can also ask about proper actions of groupoids on  $C^*$ -algebras, extending the notion of proper actions of groupoids on spaces given in [Re].

In section 1 we deal with integrable actions, while in section 2 we discuss  $C^*$ -valued weights. Section 3 is concerned with the special case of algebras of continuous functions on a locally compact space with values in a  $C^*$ -algebra. In section 4 we combine the

earlier material to define and discuss proper actions. The functoriality properties of the situation are discussed in section 5. We also show there that the commonly used structure of  $C^*$ -algebras “proper over an action on an ordinary space” [Ks, GHT] falls not only within our present context of proper actions, but, even more, within the context of [Rf7], where strong Morita equivalence of the generalized fixed-point algebra with an ideal of the crossed product algebra is established. In section 6 we discuss how one might define the generalized fixed-point algebra for our present setting. Section 7 is devoted to actions on the algebra of compact operators, and their relation to square-integrable representations. Then in section 8 we continue that discussion by considering the orthogonality relations. Substantial parts of sections 7 and 8 can be viewed as expository, treating well-known material from a slightly different point of view.

## 1. Integrable Actions.

The material discussed here is very close to material on integrable actions in the von Neumann algebra literature. See for example definition 2.1 of [CT], the introduction to [Ld], [Pa], and 18.20 of [S]. Here we stress the  $C^*$ -algebra version of integrable actions, so that we can contrast it with the notion of proper actions which we discuss in the next section. Since every proper action is integrable, this section also develops those facts about proper actions which hold because they are integrable actions.

Let  $G$  be a locally compact group, equipped with a choice of left Haar measure. Let  $\alpha$  be a (strongly continuous) action of  $G$  on a  $C^*$ -algebra  $A$ . (A simple but useful example to keep in mind during the following discussion is the case of  $G = \mathbb{R}$  acting on the one-point compactification,  $\tilde{\mathbb{R}}$ , of  $\mathbb{R}$  by translation, leaving the point at infinity fixed, and so acting on  $A = C(\tilde{\mathbb{R}})$ . But in general we do not assume that  $A$  has an identity element.) Notice that for given  $a \in A$  the function  $x \mapsto \alpha_x(a)$  has constant norm, and so can not be integrable over  $G$  (unless  $a = 0$ ) when  $G$  is not compact. Nevertheless, our aim is to give meaning to

$$\int_G \alpha_x(a) dx,$$

at least for some actions  $\alpha$  and some  $a \neq 0$ .

It is convenient initially to place this matter in a more general context. Let  $X$  be a locally compact space (e.g.  $G$ ) and fix on it a positive Radon measure (e.g. Haar measure), to which we will not give a particular symbol. Consider the  $C^*$ -algebra  $B = C_b(X, A)$  of bounded norm-continuous functions from  $X$  to  $A$ . We can surely integrate functions of compact support. But there may be other functions, even ones of constant norm, whose integrals we can make sense of indirectly.

For the case of  $G$  and  $\alpha$ , we identify  $a \in A$  with the function  $x \mapsto \alpha_x(a)$  in  $C_b(G, A)$ . This gives an isometric inclusion of  $A$  as a  $C^*$ -subalgebra of  $C_b(G, A)$  (consisting entirely of functions of constant norm), whose image we will denote by  $A_\alpha$ . So we see that it may be useful in our more general case of  $X$  to consider eventually various subalgebras of  $B$ . For example, our considerations can be applied to the induced  $C^*$ -algebras studied for instance in [QR]. Here one has both an action  $\alpha$  on  $A$  and an action,  $\tau$ , on a space

$M$ , and one considers the subalgebra of  $C_b(M, A)$  consisting of the functions  $f$  such that  $f(\tau_x^{-1}(m)) = \alpha_x(f(m))$ .

For any positive  $\lambda \in L^1(X)$  define a linear map,  $p_\lambda$ , from  $B$  to  $A$  by

$$p_\lambda(f) = \int f(x)\lambda(x)dx.$$

It is easily seen that  $p_\lambda$  is positive, in fact completely positive [KR2], and of norm  $\|\lambda\|_1$ . We would like to have the flexibility of having  $\lambda$  range over characteristic functions of compact sets, or over continuous functions which approximate them. It is thus convenient for us to set, for use throughout this paper,

$$\mathcal{B} = \mathcal{B}(X) = \{\lambda \in L^\infty(X) : \lambda \text{ has compact support and } 0 \leq \lambda \leq 1\}.$$

We note that if  $\lambda \in \mathcal{B}$  then  $\lambda \in L^1(X)$ , so that  $p_\lambda$  is defined. Also,  $\mathcal{B}$  is an upward directed set under the usual ordering of functions, and if  $\lambda_1, \lambda_2 \in \mathcal{B}$  with  $\lambda_1 \leq \lambda_2$ , then  $p_{\lambda_1} \leq p_{\lambda_2}$  for the usual ordering of positive maps. Thus  $\{p_\lambda\}_{\lambda \in \mathcal{B}}$  is an increasing net of completely positive maps from  $B$  into  $A$ . Let  $f \in B^+$  (the positive part of  $B$ ). Then  $\{p_\lambda(f)\}_{\lambda \in \mathcal{B}}$  is an increasing net of positive elements of  $A$ . Even if this net is bounded, we can not expect it to converge in  $A$ . But bounded increasing nets of positive elements do converge (for the strong operator topology) if they are in a von Neumann algebra (5.1.4 of [KR2]). Thus if we view  $p_\lambda$  as taking values in the double-dual (or “enveloping”) von Neumann algebra,  $A''$ , of  $A$  (see 3.7 of [Pe2]), we will have such convergence. Let us examine this situation a bit more generally.

**1.1 Definition.** Let  $N$  be some von Neumann algebra, and let  $P = \{p_\lambda\}$  be an increasing net of positive maps from a  $C^*$ -algebra  $B$  to  $N$ . We say that  $b \in B^+$  is  $P$ -bounded if the net  $\{p_\lambda(b)\}$  is bounded above. Let  $\mathcal{M}_P^+$  denote the set of  $P$ -bounded elements. For each  $b \in \mathcal{M}_P^+$  let  $\varphi_P(b)$  denote the least upper bound of  $\{p_\lambda(b)\}$  in  $N$ . We call the mapping  $\varphi_P$  from  $\mathcal{M}_P^+$  to  $N$  the *least upper bound* of the net  $P$ .

It is evident that  $\mathcal{M}_P^+$  is a hereditary cone in  $B^+$ , and that  $\varphi_P$  is “linear” and positive. Now for any hereditary cone  $\mathcal{M}^+$  in a  $C^*$ -algebra  $B$  we have the following structure. (See 7.5.2 of [KR2] or 5.1.2 of [Pe2].) Let  $\mathcal{N} = \{b \in B : b^*b \in \mathcal{M}^+\}$ . Then  $\mathcal{N}$  is a left ideal in  $B$  (not necessarily closed). Let  $\mathcal{M}$  be the linear span of  $\mathcal{M}^+$ . Then  $\mathcal{M} = \mathcal{N}^*\mathcal{N}$  (linear span of products), and  $\mathcal{M} \cap B^+ = \mathcal{M}^+$ . In particular,  $\mathcal{M}$  is a hereditary  $*$ -subalgebra of  $B$ . If  $\varphi$  is an additive map from  $\mathcal{M}^+$  to  $M^+$  for even some  $C^*$ -algebra  $M$ , and if  $\varphi(ra) = r\varphi(a)$  for  $r \in \mathbb{R}^+$  and  $a \in \mathcal{M}^+$ , then the usual proof for scalar-valued weights shows that  $\varphi$  has a positive linear extension to  $\mathcal{M}$ , which is unique. An important slippery point in connection with all this is that if  $a \in \mathcal{M}$ , it does not follow in general that  $|a| \in \mathcal{M}$  (even if  $a = a^*$ ). This difficulty already occurs with ordinary weights. See the example following theorem 2.4 of [Pe1]. This makes it awkward to define an “ $L^1$ -norm” on  $\mathcal{M}$  using  $\varphi$ . In Theorem 8.9 we will give a simple explicit example in which this difficulty occurs exactly in our context, namely for an integrable action on the algebra of compact operators.

Returning to our situation of an increasing net  $P = \{p_\lambda\}$  with least upper bound  $\varphi_P$ , we see that  $\varphi_P$  extends to the linear span,  $\mathcal{M}_P$ , of  $\mathcal{M}_P^+$ . Then it is clear that for  $b \in \mathcal{M}_P$  the net  $\{p_\lambda(b)\}$  converges strongly to  $\varphi_P(b)$ . It is not difficult to verify that if each  $p_\lambda$

is completely positive, then  $\varphi_P$  is also. This suggests the following definition, where we momentarily allow values in a  $C^*$ -algebra rather than a von Neumann algebra. This definition is essentially 1.1 of [Ku] with  $C = M(A)$ .

**1.2 Definition.** By a  $C^*$ -valued weight on a  $C^*$ -algebra  $B$  we mean a function,  $\varphi$ , whose domain is a hereditary cone  $\mathcal{M}^+$  in  $B^+$ , and whose range is contained in  $C^+$  for some  $C^*$ -algebra  $C$ , such that

- 1)  $\varphi(ra) = r\varphi(a)$  for  $a \in \mathcal{M}^+$  and  $r \in \mathbb{R}^+$ ,
- 2)  $\varphi(a + b) = \varphi(a) + \varphi(b)$  for  $a, b \in \mathcal{M}^+$ .

We will say that  $\varphi$  is *completely* positive if the unique positive extension of  $\varphi$  to the linear span,  $\mathcal{M}$ , of  $\mathcal{M}^+$  is completely positive, in the sense that for all  $n$ , if  $(b_{ij})$  is an  $n \times n$  matrix of elements of  $\mathcal{M}$  which is positive as an element of  $M_n(B)$ , then the matrix  $(\varphi(b_{ij}))$  is positive as a matrix in  $M_n(C)$ . If the values are in a von Neumann algebra, we will refer to  $\varphi$  as an *operator-valued weight* on  $B$ .

In the case in which  $\varphi$  comes from an increasing net of positive maps as above, with values in a von Neumann algebra  $N$ , it is natural in view of standard definitions in the literature (see 5.1.1 of [Pe2]), to make the following definition:

**1.3 Definition.** An operator-valued weight  $\varphi$  on a  $C^*$ -algebra  $B$ , with domain  $\mathcal{M}^+$  and range in  $N$ , is said to be *normal* if there is an increasing net  $\{p_\lambda\}$  of bounded positive linear maps from  $B$  into  $\mathcal{N}$  such that

- 1)  $\mathcal{M}^+ = \{b \in B^+ : \{p_\lambda(a)\} \text{ is bounded above}\}$ ,
- 2)  $\varphi(b) = \text{l.u.b.}\{p_\lambda(b)\}$  for  $b \in \mathcal{M}^+$ .

We return to the situation in which  $B = C_b(X, A)$ .

**1.4 Definition.** Let  $B = C_b(X, A)$ , and let  $P = \{p_\lambda\}$  be as defined earlier. The elements in the linear span of the  $P$ -bounded elements will be called the *order-integrable* elements of  $B$ .

This definition is closely related to Exel's definition of pseudo-integrable elements [E1, E2], the difference being that we emphasize the order structure rather than the unconditional integrability. It is different from the definition given in 7.8.4 of [Pe2]. Rather we will see that the latter is very close to our definition of *proper* actions given in the next section.

By considering the continuity of the  $p_\lambda$ 's we obtain the following alternative characterization of positive order-integrable elements in this case:

**1.5 Proposition.** *An element  $f \in B^+$  is order-integrable iff there is a constant,  $k_f$ , such that*

$$\|p_\lambda(f)\| \leq k_f \|\lambda\|_\infty$$

*for every  $\lambda \in L^\infty(G) \cap L^1(G)$  with  $\lambda \geq 0$ . (Equivalently, we can omit the condition  $\lambda \geq 0$ .)*

With a possible change in the constant  $k_f$ , we obtain:

**1.6 Corollary.** *For every order-integrable element  $f \in B$  there is a constant,  $k_f$ , such that*

$$\|p_\lambda(f)\| \leq k_f \|\lambda\|_\infty$$

*for all  $\lambda \in L^1(G) \cap L^\infty(G)$ .*

**1.7 Notation.** We will denote the hereditary \*-subalgebra of order-integrable elements by  $\mathcal{M}_X$ , and the left ideal  $\{a \in A : a^*a \in \mathcal{M}_X\}$  by  $\mathcal{N}_X$ . We denote the associated operator-valued weight with values in  $A''$ , and its unique extension to  $\mathcal{M}_X$ , by  $\varphi_X$ . It is natural to also denote  $\varphi_X(f)$  for  $f \in \mathcal{M}_X$  by

$$\varphi_X(f) = \int f(x)dx,$$

as long as the integral is interpreted as simply meaning  $\varphi_X(f)$ .

We will later find the following fact useful.

**1.8 Proposition.** *Let  $f \in \mathcal{M}_X$ , and let  $\omega \in A'$ , the dual space of  $A$ . Then the function  $x \mapsto \omega(f(x))$  is integrable (in the ordinary sense) on  $X$ , and*

$$\int \omega(f(x))dx = \omega(\varphi_X(f)),$$

*where  $\omega$  is viewed as being in the predual of  $A''$ .*

*Proof.* By the definition of  $\mathcal{M}_X$  and by the standard decomposition [Pe2] of elements of  $A'$  in terms of positive elements, it suffices to treat the case of positive  $a$  and positive  $\omega$ . The function in question is continuous, so measurable. For  $\lambda \in \mathcal{B}$  we have

$$\int \lambda(x)\omega(f(x))dx = \omega(p_\lambda(f)) \leq \|\omega\|k_f,$$

where  $k_f$  is as in Corollary 1.6. Then a short argument using the monotone convergence theorem shows that the function is integrable. The weak-\* topology on  $A''$  coincides with the ultra-weak operator topology (3.5.5-6 of [Pe2]), so the equality must hold.  $\square$

As indicated above, we are interested in subalgebras of  $B$ . The main definition of this section is:

**1.9 Definition.** Let  $B = C_b(X, A)$  as above, and let  $C$  be a  $C^*$ -subalgebra of  $B$ . We will say that  $C$  is an *integrable* subalgebra if  $C \cap \mathcal{M}_X$  is dense in  $C$ .

We remark that this definition can even be applied to operator systems, i.e. self-adjoint subspaces, and might eventually be useful there. Our main application of this definition is to the case of an action  $\alpha$  of  $G$  on  $A$ , and the subalgebra  $A_\alpha$  of  $B$ , defined above, consisting of the functions  $x \mapsto \alpha_x(a)$ . Here  $X = G$ , and we set  $\mathcal{M}_\alpha^+ = \mathcal{M}_G^+ \cap A_\alpha$ , and similarly for  $\mathcal{N}_\alpha$ ,  $\mathcal{M}_\alpha$ , and  $\varphi_\alpha$ . But we often tacitly identify  $A$  with  $A_\alpha$ .

**1.10 Definition.** The action  $\alpha$  of  $G$  on  $A$  is said to be *integrable* if  $A_\alpha$  is an integrable subalgebra of  $C_b(G, A)$ , that is, if  $\mathcal{M}_\alpha$  is dense in  $A$ .

We remark that this definition is very close to that given in the sentence before 18.20 of [S] for the setting of a Kac algebra acting on a von Neumann algebra. The case of a group acting on a von Neumann algebra is then discussed in 18.20 of [S].

A question which I have not been able to resolve is whether, given an integrable action  $\alpha$  of  $G$  on  $A$ , and given a closed subgroup  $H$  of  $G$ , the restriction of  $\alpha$  to  $H$  must always be integrable. This question is closely related to the notion of “strongly subgroup integrable” introduced in definition 2.17 of [Rf6] and discussed there.

The following observation about integrable actions is motivated by well-known considerations in topological dynamics concerning wandering sets. (See Theorem 6.15 of [W].)

**1.11 Proposition.** *Let  $\alpha$  be an action of  $G$  on  $A$ . Suppose that  $G$  is not compact. Then every  $\alpha$ -invariant state on  $A$  has value 0 on all of  $\mathcal{M}_\alpha$ . In particular, if  $\alpha$  is integrable then there are no  $\alpha$ -invariant states on  $A$ .*

*Proof.* Let  $\omega$  be an  $\alpha$ -invariant state on  $A$ , and let  $a \in \mathcal{M}_\alpha$ . By Proposition 1.8 the function  $x \mapsto \omega(\alpha_x(a)) = \omega(a)$  must be integrable. Since  $G$  is not compact,  $\omega(a) = 0$ .  $\square$

We now return to the case of a general action  $\alpha$ . By transport of structure, each  $\alpha_x$  extends to an automorphism of  $A''$ , still denoted by  $\alpha_x$ , though the corresponding action of  $G$  will usually not be continuous for the norm. We will let  $(A'')^\alpha$  denote the fixed point subalgebra of  $A''$  for this action.

Now for any  $a \in \mathcal{M}_\alpha^+$  and  $x \in G$  the net  $\{\alpha_x(\alpha_f(a))\}_{f \in \mathcal{B}}$  must have  $\alpha_x(\varphi(a))$  as l.u.b. Let  $L_x$  denote the usual left translation operator on functions on  $G$  defined by  $(L_x f)(y) = f(x^{-1}y)$ . Then for  $f \in L^1(G)$  we have  $\alpha_x \alpha_f = \alpha_{L_x f}$ . Furthermore,  $L_x$  on  $\mathcal{B}$  is clearly an order automorphism of  $\mathcal{B}$ . Thus the l.u.b. of  $\{\alpha_x(\alpha_f(a))\}$  must still be  $\varphi(a)$ . Consequently  $\alpha_x(\varphi(a)) = \varphi(a)$ . We have thus obtained:

**1.12 Proposition.** *The operator-valued weight  $\varphi_\alpha$  has values in  $(A'')^\alpha$ .*

We now examine the action of  $\alpha$  on elements of  $\mathcal{M}_\alpha$ . Let  $\Delta$  denote the modular function of  $G$ , with the convention that

$$\int f(xy)dx = \Delta(y^{-1}) \int f(x)dx, \quad \int f(x^{-1})dx = \int f(x)\Delta(x^{-1})dx.$$

For  $x \in G$  let  $R_x$  be the operator of right translation on functions on  $G$  defined by  $(R_x f)(y) = f(yx^{-1})$ . We choose this convention both because then  $R_x$  is an order automorphism of  $\mathcal{B}$  (not necessarily preserving the  $L^1$ -norm), and because a simple calculation shows that for  $f \in L^1(G)$  and  $a \in A$  we have

$$\alpha_f(\alpha_x(a)) = \Delta(x)^{-1} \alpha_{R_x f}(a).$$

Arguing as we did for Proposition 1.12 we obtain:

**1.13 Proposition.** *Let  $a \in \mathcal{M}_\alpha$ . Then  $\alpha_x(a) \in \mathcal{M}_\alpha$  for any  $x \in G$ , and*

$$\varphi_\alpha(\alpha_x(a)) = \Delta(x)^{-1} \varphi_\alpha(a).$$

**1.14 Corollary.** *The left ideal  $\mathcal{N}_\alpha$  is carried into itself by  $\alpha$ .*

*Proof.* If  $a \in \mathcal{N}_\alpha$  then  $a^*a \in \mathcal{M}_\alpha$ , so that  $\alpha_x(a)^*\alpha_x(a) = \alpha_x(a^*a) \in \mathcal{M}_\alpha$  for any  $x \in G$ .  $\square$

It is natural to define an  $(A'')^\alpha$ -valued inner-product on  $\mathcal{N}_\alpha$  by

$$\langle a, b \rangle_\alpha = \varphi_\alpha(a^*b).$$

Note that  $\mathcal{N}_\alpha$  will not in general be a right module over  $(A'')^\alpha$ , so that  $\mathcal{N}_\alpha$  need not be a Hilbert  $C^*$ -module. One can extend  $\mathcal{N}_\alpha$  to get a Hilbert  $C^*$ -module by passing to a suitable von Neumann subalgebra of  $A''$ , but we will not pursue this matter here. In any case, we do have a corresponding norm on  $\mathcal{N}_\alpha$  defined by  $\|\langle a, a \rangle_\alpha\|^{1/2}$ , where the norm in this expression is that of  $A''$ .

**1.15 Notation.** For  $x \in G$  we define an operator,  $U_x$ , on  $\mathcal{N}_\alpha$  by

$$U_x a = \Delta(x)^{1/2} \alpha_x(a).$$

A simple calculation shows that  $U_x$  is “unitary” in the sense that

$$\langle U_x a, U_x b \rangle_\alpha = \langle a, b \rangle_\alpha$$

for  $a, b \in \mathcal{N}_\alpha$ . We obtain in this way a group homomorphism from  $G$  into the group of “unitary” operators on  $\mathcal{N}_\alpha$ . It is not clear to me how often this homomorphism will be strongly continuous for the norm defined above. This seems to be quite a delicate matter to ascertain in various examples. This is closely related to:

**1.16 Question.** Under what circumstances will it be true that for every finite measure  $\mu$  of compact support on  $G$  we have  $\alpha_\mu(a) \in \mathcal{N}_\alpha$  if  $a \in \mathcal{N}_\alpha$  (where  $\alpha_\mu$  is the integrated form of  $\alpha$ )?

We can show that this is true for  $a \in \mathcal{M}_\alpha$ , but we will not pursue this matter here.

For use in the next section we now examine to some extent what integrability means in the commutative case.

**1.17 Proposition.** *Let  $\alpha$  be an action of  $G$  on the locally compact space  $M$ , and so on the  $C^*$ -algebra  $C_\infty(M)$  of functions vanishing at infinity. If  $\alpha$  on  $C_\infty(M)$  is integrable, then every  $\alpha$ -orbit in  $G$  is closed, and the stability subgroup of each point of  $M$  is compact.*

*Proof.* Suppose the  $\alpha$ -orbit of  $m_0 \in M$  is not closed, so that it has a limit point  $n$  which is not in the orbit. Choose  $f \in C_c(M)^+$  (functions of compact support) such that  $f(n) > 1$ . Let  $U = \{m \in M : f(m) > 1\}$ , so that  $U$  is a neighborhood of  $n$ . By the joint continuity of the action, we can find a symmetric open precompact neighborhood  $\mathcal{O}$  of  $e_G$  (the identity element of  $G$ ) and a neighborhood  $V$  of  $n$  such that  $\alpha_{\mathcal{O}}(V) \subset U$ . Choose a sequence  $\{x_j\}$  in  $G$  by induction as follows. Set  $x_1 = e_G$  (the identity element). If  $x_1, \dots, x_k$  have been chosen, let  $C_k$  be the closure of the union of the  $x_j^{-1}\mathcal{O}^2$  for  $j \leq k$ . Then  $C_k$  is a compact set. Thus  $\alpha_{C_k}(m_0)$  is a compact subset of the orbit of  $m_0$ , and so can not have  $n$  in its closure. Thus we can find  $x_{k+1} \in G$  such that  $x_{k+1}^{-1} \notin C_k$  and  $\alpha_{x_{k+1}}^{-1}(m_0) \in V$ .

Notice that since  $\mathcal{O}$  is symmetric, all the sets  $x_j^{-1}\mathcal{O}$  are disjoint. But if  $y \in \mathcal{O}$ , then

$$\alpha_{(x_j y)}^{-1}(m_0) = \alpha_y^{-1} \alpha_{x_j}^{-1}(m_0) \in U,$$

so that

$$(\alpha_{x_j y} f)(m_0) \geq 1.$$

That is, the function  $x \mapsto (\alpha_x(f))(m_0)$  is non-negative, and has value  $\geq 1$  on each of the disjoint sets  $x_n \mathcal{O}$ , which all have the same non-zero Haar measure. Thus this function can not be integrable. If we view evaluation at  $m_0$  as a continuous linear functional on  $A$ , then from Proposition 1.8 it follows that  $f \notin \mathcal{M}_\alpha^+$ . But  $\mathcal{M}_\alpha$  is a hereditary  $*$ -subalgebra, which in the commutative case means an ideal. Thus if  $\mathcal{M}_\alpha$  were dense it would have to contain  $C_c(M)$ . Thus  $\alpha$  is not integrable.

We now show that stability groups are compact. Let  $m \in M$ , and pick  $f \in C_c(M)^+$  such that  $f(m) > 1$ . Let  $g(x) = f(\alpha_x^{-1}(m))$ . Then there is a compact symmetric neighborhood  $\mathcal{O}$  of  $e_G$  on which  $g \geq 1$ . Let  $G_m$  denote the stability subgroup of  $m$ . Then  $g(xs) = g(x)$  for  $x \in G$ ,  $s \in G_m$ . If  $G_m$  is not compact, it is easily seen from this that  $g$  can not be integrable.  $\square$

I have not noticed simple conditions which are simultaneously necessary and sufficient for  $\alpha$  to be integrable. It is not sufficient just to have the orbits be closed and the stability subgroups be compact. This is seen by the following example, which is a slightly more complicated version of the example at the very end of Philip Green's article [G]. We make our example yet slightly more complicated than needed here so that we can also use it in the next section to illustrate a point there.

**1.18 Example.** The space  $M$  is a closed subset of  $\mathbb{R}^3$ , and the group  $G$  is  $\mathbb{R}$ . The action is free, with all orbits closed. The orbit space  $M/\alpha$  is a compact Hausdorff space consisting of a countable number of points, which is discrete except for one limit point. This limit orbit is the “ $y$ -axis”  $\{(0, s, 0) : s \in \mathbb{R}\}$ , with the action  $\alpha$  of  $\mathbb{R}$  on it being by translation. We denote this orbit by  $\mathcal{O}_*$ , and we let  $p_* = (0, 0, 0)$ , which is one of its points. We label the other orbits by strictly positive integers,  $n$ , and we denote the  $n$ -th orbit by  $\mathcal{O}_n$ . Part of the data specifying these orbits consists of a strictly decreasing sequence  $\{b_n\}$  of real numbers which converges to 0. Let  $p_n = (b_n, 0, 0)$ . Then  $p_n$  will be in  $\mathcal{O}_n$ . Up to equivariant homeomorphism the example will be independent of the choice of  $\{b_n\}$ . However, it does depend on the next piece of data, which is an assignment to each  $n$  of a strictly positive integer,  $L_n$ , which should be thought of as a “repetition number”. However, the example will be independent of the choice of the next piece of data, which is an assignment to each  $n$  of a strictly decreasing finite sequence  $\{c_j^n\}$  of length  $L_n$ , with  $b_{n+1} < c_j^n < b_n$ . Let  $q_j^n = (c_j^n, 0, 0)$ . Each of the points  $q_j^n$ ,  $j = 1, \dots, L_n$ , will be in the orbit  $\mathcal{O}_n$ . At one place below it is convenient to set  $c_0^n = b_n$ . We specify  $\mathcal{O}_n$  and the action  $\alpha$  by saying where  $\alpha$

takes  $p_n$ . For  $t \in \mathbb{R}$  we set

$$\alpha_t(p_n) = \begin{cases} (b_n, t, 0) & t \in (-\infty, n]. \\ (c_{L_n}^n, s, 0) & s \in (-n, +\infty), t = s + L_n(2n+1). \\ (c_j^n, s, 0) & s \in (-n, n], t = s + j(2n+1), \\ & 1 \leq j < L_n, \text{ void if } L_n = 1. \\ ((1-s)c_j^n + sc_{j+1}^n, n \cos(\pi s), n \sin(\pi s)) & s \in (0, 1], t = s + n + j(2n+1), \\ & 0 \leq j \leq L_n - 1, c_0^n = b_n. \end{cases}$$

If one contemplates the facts that whenever points of  $m$  are in the  $x$ - $y$ -plane and not about to leave it they move parallel to the  $y$ -axis with unit speed, and that, as  $n$  increases, the  $y$ -coordinates where points enter and leave the  $x$ - $y$ -plane go to  $-\infty$  and  $+\infty$ , one sees that this action is indeed jointly continuous, and that the properties stated at the beginning are satisfied. In particular, for any  $f \in C_c(M)$ , the support of  $f$  will meet any given orbit in a compact set, and thus

$$\int_G f(\alpha_x(m)) dx$$

will be finite for each  $m \in M$ .

Choose now an  $f \in C_c(\mathcal{O}_*)^+$ , supported strictly inside  $\{0\} \times [-1/2, 1/2] \times \{0\}$ , and such that its integral over  $\mathcal{O}_*$ , i.e.  $\int f(\alpha_{-t}(p_*)) dt = 1$ . Extend  $f$  to a function in  $C_c(\mathbb{R}^2 \times \{0\})^+$ , still denoted by  $f$ , in such a way that the support of  $f$  is contained in the disk of radius  $1/2$  about the origin, and that for some  $\varepsilon > 0$  this extended  $f$  is independent of the  $x$ -coordinate. We restrict  $f$  to  $M$  and still denote it by  $f$ . Then as soon as  $n$  is large enough that  $b_n < \varepsilon$ , the restriction of  $f$  to  $\mathcal{O}_n$  will, on  $G$ , look like  $L_n + 1$  copies of  $f$  with disjoint supports. Consequently, for each large  $n$  we have

$$\int f(\alpha_{-t}(p_n)) dt = L_n + 1.$$

In other words, for large enough  $n$ ,

$$\int \alpha_t(f) dt = \begin{cases} 1 & \text{on } \mathcal{O}_* \\ L_n + 1 & \text{on } \mathcal{O}_n. \end{cases}$$

In particular, if the sequence  $\{L_n\}$  is not bounded, then  $f$  is not integrable, so that  $\mathcal{M}_\alpha$  is not dense, and so the action  $\alpha$  is not integrable. On the other hand, if the sequence  $\{L_n\}$  is bounded, then one can check that  $\alpha$  is integrable.

We remark that by examining the foliations of the plane (which come from actions of  $\mathbb{R}$ ), as studied in [Wn], we obtain an abundance of examples of integrable actions  $\alpha$  on  $C_\infty(M)$  such that  $M/\alpha$  is not Hausdorff (but the actions are free, with closed orbits). Thus, integrability does not imply that  $M/\alpha$  is Hausdorff.

Anyway, we are left with:

**1.19 Question.** What are conditions just in terms of an action  $\alpha$  of a group  $G$  on a space  $M$  which are simultaneously necessary and sufficient for  $\alpha$  on  $C_\infty(M)$  to be integrable?

We conclude this section with an important property of integrable actions with respect to tensor products. Many special cases of this property are employed in the literature. (See e.g. 18.21 of [S].)

**1.20 Proposition.** *Let  $\alpha$  and  $\beta$  be actions of  $G$  on the  $C^*$ -algebras  $A$  and  $B$ , and let  $\alpha \otimes \beta$  denote the corresponding action on their maximal, or minimal, tensor product,  $A \otimes B$ . If  $\alpha$  is integrable, then so is  $\alpha \otimes \beta$ .*

*Proof.* It does not hurt to adjoin a unit to  $B$  if it does not have one. So we assume that  $B$  is unital. Let  $a \in \mathcal{M}_\alpha^+$ . Then

$$(\alpha \otimes \beta)_\lambda(a \otimes 1_B) = \alpha_\lambda(a) \otimes 1 \leq k_a \|\lambda\|_\infty$$

with our earlier notation. Thus  $a \otimes 1_B \in \mathcal{M}_{\alpha \otimes \beta}^+$ . But for any  $b \in B^+$  we have  $a \otimes b \leq \|b\|(a \otimes 1_B)$ . Since  $\mathcal{M}_{\alpha \otimes \beta}^+$  is hereditary, it follows that  $a \otimes b \in \mathcal{M}_{\alpha \otimes \beta}^+$ . Since  $\mathcal{M}_\alpha$  is dense in  $A$  by assumption, it follows that  $\mathcal{M}_{\alpha \otimes \beta}$  is dense in  $A \otimes B$ .  $\square$

## 2. Strict $C^*$ -weights.

We recall that if  $\alpha$  is an action of  $G$  on a locally compact space  $M$ , then  $\alpha$  is said to be *proper* if the map  $(m, x) \rightarrow (m, \alpha_x(m))$  from  $M \times G$  to  $M \times M$  is proper, in the sense that preimages of compact sets are compact. It is well known [Bo] that in this case the orbit space,  $M/\alpha$ , with the quotient topology from  $M$ , is locally compact Hausdorff. The functions in  $C_\infty(M/\alpha)$  can be viewed as functions in  $C_b(M)$  which are  $\alpha$ -invariant. Here  $C_b(M)$  is the algebra of bounded continuous functions on  $M$ , and it is the multiplier algebra of  $C_\infty(M)$ . It is well-known (2.4 of [Pn]) that if  $h \in C_c(M)$ , and if we set

$$\psi(h)(m) = \int h(\alpha_x^{-1}(m)) dx$$

for every  $m \in M$ , then  $\psi(h)$  is a function in  $C_\infty(M/\alpha) \subseteq C_b(M)$ . It is natural to write

$$\psi(h) = \int \alpha_x(h) dx,$$

but as before, the integrand is not integrable in the usual sense if  $G$  is not compact. But if we consider  $\alpha_\lambda(h)$  for  $\lambda \in \mathcal{B}$  as in the previous section, it is easily seen that  $\alpha_\lambda(h)$  converges to  $\psi(h)$  in the strict topology, that is,  $k\alpha_\lambda(h)$  converges to  $k\psi(h)$  in (uniform) norm for every  $k \in C_\infty(M)$ . See the discussion early in section 1 of [Rf7]. For the definition and basic properties of the strict topology see [La, Pe2].

There are a number of situations in which an action of a group on a non-commutative  $C^*$ -algebra seems to be proper in some sense. I tried to give an appropriate definition in [Rf7]. The definition given there was adequate to treat some interesting examples, but it assumed the existence of a dense subalgebra with certain properties, and so was not intrinsic. I propose here a tentative intrinsic definition, which is essentially one almost explicitly suggested by Exel in section 6 of [E1]. The difference is that here we emphasize the order properties, paralleling the development in our first section, while Exel emphasizes unconditional integrability.

As suggested by the above discussion, this matter leads to weights with values in  $C^*$ -algebras. We mentioned in the previous section that such weights have recently been

introduced by Kustermans [Ku] for use in connection with quantum groups (though so far I have not seen how to use his “regularity” condition in the present context). Much as we need here, he treats weights on a  $C^*$ -algebra  $B$  with values in  $M(A)$  for another  $C^*$ -algebra  $A$ . (Here  $M(A)$  denotes the multiplier algebra [Pe2] of  $A$ .) Our basic context is as follows:

**2.1 Definition.** Let  $B$  and  $A$  be  $C^*$ -algebras, and let  $P = \{p_\lambda\}$  be an increasing net of positive operators from  $B$  into  $M(A)$ . We say that  $b \in B^+$  is  $P$ -proper if the net  $\{p_\lambda(b)\}$  converges in the strict topology to an element,  $\psi_P(b)$ , of  $M(A)$ . We denote the set of  $P$ -proper elements of  $B^+$  by  $\mathcal{P}_P^+$ .

It is clear that  $\mathcal{P}_P^+$  is a cone, and that  $\psi_P$  is “linear” on  $\mathcal{P}_P^+$ .

For use in dealing with this definition we now recall several of the basic facts about the strict topology which Kustermans obtains, in a form suitable for our needs here. The considerations here parallel somewhat the strong and weak operator topologies. A small novelty is our explicit definition of the “weak-strict” topology. (It has been used implicitly in earlier work.)

**2.2 Definition.** We say that a net  $\{m_\lambda\}$  in  $M(A)$  converges in the *weak-strict topology* to  $n \in M(A)$  if the net  $\{am_\lambda c\}$  converges in norm to  $anc$  for every  $a, c \in A$ . By polarization it suffices to consider  $\{a^*m_\lambda a\}$  and  $a^*na$ . We will say that a net  $\{m_\lambda\}$  is *weak-strict Cauchy* if for every  $a, c \in A$  the net  $\{am_\lambda c\}$  in  $A$  is norm-Cauchy. Again, it suffices to examine the nets  $\{a^*m_\lambda a\}$  for  $a \in A$ .

It is clear that if a net is strictly convergent, or is strictly Cauchy, then it is also so weak-strictly.

**2.3 Proposition.** Let  $\{m_\lambda\}$  be an increasing net in  $M(A)^+$  which converges weak-strictly to  $n \in M(A)$ . Then  $m_\lambda \leq n$  for all  $\lambda$ . In particular,  $\{m_\lambda\}$  is bounded in norm.

*Proof.* Fix any  $\lambda_0$ , and set  $k = n - m_{\lambda_0}$  and  $k_\lambda = m_\lambda - m_{\lambda_0}$ . Then the net  $\{k_\lambda\}$  is eventually positive and converges weak-strictly to  $k$ . Thus for any  $a \in A$  the net  $\{a^*k_\lambda a\}$  is eventually positive and converges in norm to  $a^*ka$ . Thus  $a^*ka$  is positive for all  $a \in A$ . Then it is not hard to see that  $k$  is positive. (See Lemma 4.1 of [La].)  $\square$

In lemma 9.3 of [Ku] Kustermans uses the uniform boundedness principle several times to show that in the above proposition it suffices to assume that  $\{m_\lambda\}$  is weak-strict Cauchy. This observation can be useful in connection with the following proposition.

**2.4 Proposition.** (See 9.4 and 9.5 of [Ku].) Let  $\{m_\lambda\}$  be an increasing net in  $M(A)^+$  which is weak-strict Cauchy and is bounded in norm. Then  $\{m_\lambda\}$  is strictly Cauchy, and so converges strictly (and so weak-strictly) to an element of  $M(A)^+$ .

*Proof.* Let  $K$  be a bound on  $\{m_\lambda\}$ . Then for  $a \in A$  and  $\lambda > \mu$  we have

$$\begin{aligned} \|(m_\lambda - m_\mu)a\|^2 &\leq \|(m_\lambda - m_\mu)^{1/2}\|^2 \|(m_\lambda - m_\mu)^{1/2}a\|^2 \\ &\leq K \|a^*(m_\lambda - m_\mu)a\|. \end{aligned}$$

Thus  $\{m_\lambda a\}$  is norm-Cauchy. By taking adjoints we see that  $\{am_\lambda\}$  too is norm-Cauchy. Thus  $\{m_\lambda\}$  is strictly Cauchy, and so converges to a positive element of  $M(A)$ , since  $M(A)$  is strictly complete [La].  $\square$

**2.5 Proposition.** (Lemma 9.4 of [Ku].) Let  $\{m_\lambda\}$  be a net of elements of  $M(A)^+$ , and let  $m \in M(A)^+$  be such that  $m_\lambda \leq m$  for all  $\lambda$ . If  $\{m_\lambda\}$  converges weak-strictly to  $m$ , then it does so strictly.

*Proof.* For any  $a \in A$  we have, as above,

$$\begin{aligned} \|(m - m_\lambda)a\|^2 &\leq \|(m - m_\lambda)^{1/2}\|^2 \|(m - m_\lambda)^{1/2}a\|^2 \\ &\leq \|m\| \|a^*(m - m_\lambda)a\|, \end{aligned}$$

For  $a$  on the other side, take adjoints. □

There is a useful alternate characterization of the  $P$ -proper elements in terms of linear functionals on  $A$ . It is related to the definition of  $\hat{a}$ -integrable elements given on page 269 of [Pe2], which originated in [OP1, OP2]. But we call attention to the note at the end of [OP2] which points out that the definition in [Pe2] is too strong, since it should only consider the dual,  $B'$ , of  $B$  (in the notation of [Pe2]), not of  $M(B)$ . For some later variants see [QR] and its references. We will use the fact that each element of  $A'$  has a canonical extension to  $M(A)$ , obtained by viewing  $A'$  as the predual of  $A''$  and  $M(A)$  as canonically embedded in  $A''$  (proposition 3.12.3 of [Pe2]).

**2.6 Theorem.** Let  $P = \{p_\lambda\}$  be an increasing net of positive operators from  $B$  to  $M(A)$ , and let  $b \in B^+$ . Then  $b$  is  $P$ -proper if and only if there is an  $m \in M(A)^+$  such that for every positive linear functional,  $\omega$ , on  $A$ , viewed also as a positive linear functional on  $M(A)$ , the net  $\{\omega(p_\lambda(b))\}$  converges to  $\omega(m)$ .

*Proof.* Suppose that  $b$  is  $P$ -proper. For any bounded linear functional  $\omega$  on  $A$  and any bounded net  $\{n_\lambda\}$  in  $M(A)$  which converges strictly to  $m \in M(A)$  the net  $\{\omega(n_\lambda)\}$  converges to  $\omega(m)$ . This follows from the fact that  $\omega$  can be approximated in norm by linear functionals of the form  $\omega_u$  defined by  $\omega_u(a) = \omega(u^*au)$  for  $u \in A$ . (See e.g. the proof of theorem 3.12.9 of [Pe2].) From this it follows that if  $b$  is  $P$ -proper, then  $\{\omega(p_\lambda(b))\}$  converges to  $\omega(\psi_P(b))$ .

Suppose conversely that there is an  $m \in M(A)^+$  as in the statement of the theorem. For any positive  $\omega$  and any  $c \in A$  let  $\omega_c$  be defined by  $\omega_c(a) = \omega(c^*ac)$ . Since  $\omega_c$  is positive,  $\omega_c(p_\lambda(b))$  converges by hypothesis to  $\omega_c(m)$ , that is,  $\omega(c^*p_\lambda(b)c)$  converges to  $\omega(c^*mc)$ . But now  $c^*mc \in A$ . Let  $Q(A)$  denote the quasi-state space [Pe2] of  $A$ , consisting of those positive  $\omega \in A'$  such that  $\|\omega\| \leq 1$ . Note that  $Q(A)$  is weak-\* compact. For any  $d \in A$  let  $\hat{d}$  denote  $d$  viewed as a function on  $Q(A)$ , so that  $\hat{d}$  is continuous (and affine). With this notation,  $(c^*p_\lambda(b)c)^\wedge$  is an increasing net of continuous positive functions on  $Q(A)$ , which as we saw above converges pointwise to the continuous function  $(c^*mc)^\wedge$ . By Dini's theorem it follows that the convergence is uniform. But  $Q(A)$  determines the norm of elements of  $A^+$ . It follows that the net  $\{c^*p_\lambda(b)c\}$  converges in norm to  $c^*mc$ , that is,  $\{p_\lambda(b)\}$  converges weak-strictly to  $m$ . From Proposition 2.5 it follows that  $\{p_\lambda(b)\}$  converges to  $m$  strictly. Thus  $b$  is  $P$ -proper as desired. □

The following lemma is motivated by, and very closely related to, proposition 6.6 of [E1] and to the comments in 7.8.4 of [Pe2] and 2.4 of [OP1]. See also lemma 3.5 of [Qg] and lemma 4.1 of [Ku].

**2.7 Key Lemma.** *Let  $P = \{p_\lambda\}$  be an increasing net of positive operators from  $B$  to  $M(A)$ . Then the cone  $\mathcal{P}_P^+$  of  $P$ -proper elements of  $B^+$  is hereditary.*

*Proof.* Let  $b \in \mathcal{P}_P^+$ , and let  $b_0 \in B$  with  $0 \leq b_0 \leq b$ . The net  $\{p_\lambda(b)\}$  is bounded above by Proposition 2.3, and so the net  $\{p_\lambda(b_0)\}$  is bounded above. Since the latter net is increasing, to show that  $b_0 \in \mathcal{P}_P^+$  it suffices by Proposition 2.4 to show that  $\{p_\lambda(b_0)\}$  is weak-strict Cauchy. Now for any  $a \in A$  and any  $\lambda > \mu$

$$a^*(p_\lambda(b_0) - p_\mu(b_0))a = a^*((p_\lambda - p_\mu)(b_0))a \leq a^*((p_\lambda - p_\mu)(b))a.$$

But  $\{p_\lambda(b)\}$  converges strictly, and so is weak-strict Cauchy.  $\square$

According to the properties of hereditary cones given in section 1, if we set

$$\mathcal{Q}_P = \{b \in B : b^*b \in \mathcal{P}_P\},$$

then  $\mathcal{Q}_P$  is a left ideal in  $B$ . Let  $\mathcal{P}_P$  denote the linear span of  $\mathcal{P}_P^+$ . Then  $\mathcal{P}_P = \mathcal{Q}_P^* \mathcal{Q}_P$ ,  $\mathcal{P}_P \cap B^+ = \mathcal{P}_P^+$ , and  $\psi_P$  extends uniquely to a positive linear map from  $\mathcal{P}_P$  into  $M(B)$ .

It is clear that every  $P$ -proper positive element is  $P$ -bounded. Thus  $\mathcal{P}_P^+ \subseteq \mathcal{M}_P$  in the notation of the previous section. Furthermore, since any non-degenerate representation of  $B$  extends uniquely to one of  $M(B)$ , under which a strictly convergent net is strong operator convergent,  $\psi_P$  will be the restriction of the weight  $\varphi_P$  to the  $P$ -proper elements. The above considerations suggest:

**2.8 Definition.** Let  $A$  and  $B$  be  $C^*$ -algebras. By a  $C^*$ -valued weight on  $B$  towards  $A$  we mean a  $C^*$ -weight  $\psi$  on  $B$  (Definition 1.2) with values in  $M(A)$ . Let  $\mathcal{P}^+$  denote the domain of  $\psi$ . We say that  $\psi$  is *lower semi-continuous* if whenever  $b \in \mathcal{P}^+$  and  $\{b_\mu\}$  is an increasing net in  $B^+$  which converges in norm to  $b$ , then the net  $\{\psi(b_\mu)\}$  converges strictly to  $\psi(b)$ . We say that  $\psi$  is *strict* if there is an increasing net  $\{p_\lambda\}$  of bounded positive maps from  $B$  to  $M(A)$  for which

- 1)  $\mathcal{P}^+ = \{b \in B^+ : \text{the net } \{p_\lambda(b)\} \text{ is strictly Cauchy}\}.$
- 2)  $\psi(b) = \text{strict-lim}\{p_\lambda(b)\}$  for  $b \in \mathcal{P}^+.$

If  $B = A$  we will just say that  $\psi$  is a *lower semi-continuous*, or *strict*,  $C^*$ -valued weight on  $A$ . If  $\psi(b) = 0$  only when  $b = 0$ , we say that  $\psi$  is *faithful*.

We remark that we do not require that  $\mathcal{P}$  be dense, unlike definition 3.2 of [Ku]. Nor do we require *complete* positivity. (We will assume it explicitly when we need it.). The exact relationship between our definition of “strict” weights and Kustermans’ definition of lower semi-continuous weights in definition 3.2 of [Ku] remains to be worked out. We note that in [Ku] each  $p_\lambda$  is required to be “strict” as defined in [La]. This has some technical advantages, but in Proposition 4.11 we will see that we have an even stronger property in our group-action case.

**2.9 Proposition.** *(Basically 3.5 of [Ku].) Any strict  $C^*$ -valued weight from  $B$  toward  $A$  is lower semi-continuous.*

*Proof.* Let  $b$  and  $\{b_\mu\}$  be as in the definition above of lower semi-continuity. By Proposition 2.5 it suffices to show weak-strict convergence. Let  $a \in A$ , and let  $\varepsilon > 0$  be given. Choose  $\lambda$  so that

$$\|a\psi(b)a^* - ap_\lambda(b)a^*\| < \varepsilon/2.$$

Choose  $\mu_0$  such that if  $\mu > \mu_0$  then

$$\|ap_\lambda(b)a^* - ap_\lambda(b_\mu)a^*\| < \varepsilon/2.$$

Since we have

$$ap_\lambda(b_\mu)a^* \leq a\psi(b_\mu)a^* \leq a\psi(b)a^*,$$

it follows that for  $\mu > \mu_0$  we have

$$\|a\psi(b_\mu)a^* - a\psi(b)a^*\| < \varepsilon.$$

□

We remark that in the situation described above we can not expect that  $\psi(b_\lambda)$  will converge to  $\psi(b)$  in norm. A simple example, which we will use again later, goes as follows:

**2.10 Example.** Let  $G = \mathbb{Z}$  act by translation,  $\tau$ , on itself, and so on  $C_\infty(\mathbb{Z})$ . For each  $n \geq 1$  choose  $f_n \in C_c^+(\mathbb{Z})$  such that  $\|f_n\|_\infty \leq 1/n$  but  $\sum_k \tau_k(f_n) = 1$  strictly in  $M(C_\infty(\mathbb{Z}))$ . Let  $\alpha$  be the (proper) action of  $\mathbb{Z}$  by translation in the first variable on  $\mathbb{Z} \times \mathbb{N}$ , and so on  $B = C_\infty(\mathbb{Z} \times \mathbb{N})$ . Let  $P = \{p_\lambda\}$  come from  $\alpha$  as at the beginning of Section 1, with corresponding  $\psi$ . Let  $g \in B$  be defined by  $g(m, n) = f_n(m)$ . It is easily seen that  $g$  is  $P$ -proper, and that  $\psi(g) = 1$ . Define  $g_j$  to agree with  $g$  for  $n \leq j$  and have value 0 otherwise. Then the increasing sequence  $\{g_j\}$  converges to  $g$  in norm, while  $\{\psi(g_j)\}$  converges to  $\psi(g)$  strictly, but not in norm.

### 3. The case of $C_b(X, A)$ .

Throughout this section we let  $B = C_b(X, A)$ , and we assume that we have a positive Radon measure specified on  $X$ , in terms of which the  $p_\lambda$ 's are defined as near the beginning of Section 1. We now denote the hereditary cone of  $P$ -proper elements by  $\mathcal{P}_X^+$ , with corresponding  $\psi_X$ ,  $\mathcal{P}_X$  and  $\mathcal{Q}_X$ . We consider here some aspects which are special to this situation.

As discussed in the previous section,  $\mathcal{Q}_X$  will always be a left ideal in  $B$ . We now consider action on the right. Let  $f \in \mathcal{Q}_X$  and  $m \in M(A)$ , so that  $fm \in B$ . For  $\lambda \in \mathcal{B}$  we have

$$p_\lambda((fm)^*fm) = \int m^*f(x)^*f(x)m\lambda(x)dx = m^*p_\lambda(f^*f)m,$$

and we know that  $a^*m^*p_\lambda(f^*f)ma$  converges up in norm to  $a^*m^*\psi(f^*f)ma$ . Thus  $p_\lambda((fm)^*(fm))$  converges weak-strictly to  $m^*\psi(f^*f)m$ , and so converges strictly by Proposition 2.5. By definition it follows that  $fm \in \mathcal{Q}_X$ , so that  $\mathcal{Q}_X$  is a right  $M(A)$ -module, and

$$\psi_X(m^*f^*fm) = m^*\psi_X(f^*f)m.$$

Let also  $g \in \mathcal{Q}_X$ . Then  $g^*fm \in \mathcal{P}_X$ , and so  $\psi(g^*fm)$  is the strict limit of  $\{p_\lambda(g^*fm)\}$ . Consequently for  $a \in A$  we have the norm limits

$$a\psi_X(g^*fm) = \lim ap_\lambda(g^*fm) = \lim(ap_\lambda(g^*f))m = a\psi_X(g^*f)m.$$

Thus  $\psi_X(g^*fm) = \psi_X(g^*f)m$ . Finally, since every element of  $\mathcal{P}_X$  is a finite linear combination of elements of form  $g^*f$  for  $f, g \in \mathcal{Q}_X$ , we see that we have obtained:

**3.1 Proposition.** *Both  $\mathcal{Q}_X$  and  $\mathcal{P}_X$  are right  $M(A)$ -modules, and*

$$\psi_X(fm) = \psi_X(f)m$$

*for  $f \in \mathcal{P}_X$  and  $m \in M(A)$ .*

We can now define an  $M(A)$ -valued inner-product on  $\mathcal{Q}_X$  by

$$\langle f, g \rangle_X = \psi(f^*g) .$$

This evidently makes  $\mathcal{Q}_X$  into a right  $C^*$ -module over  $M(A)$ . (See, e.g. [La] for the definition.) Consequently we have the following version of the Cauchy-Schwarz inequality (proposition 2.9 of [Rf2], or proposition 1.1 of [La]):

**3.2 Proposition.** *For  $f, g \in \mathcal{Q}_X$  we have*

$$\psi(f^*g)^*\psi(f^*g) \leq \|\psi(f^*f)\|\psi(g^*g)$$

*in  $M(A)$ . Consequently the expression  $\|f\|_X = \|\langle f, f \rangle_X\|^{1/2}$  defines a norm on  $\mathcal{Q}_X$ .*

We will later have use for the following technical consequence.

**3.3 Proposition.** *With notation as above, for any  $\lambda \in \mathcal{B}$  and any  $f, g \in \mathcal{Q}_X$  we have*

$$\|p_\lambda(f^*g)\| \leq 4\|\langle f, f \rangle_X\| \|\langle g, g \rangle_X\| .$$

PROOF. By polarization

$$\begin{aligned} 4\|p_\lambda(f^*g)\| &\leq \sum_{k=0}^3 \|p_\lambda((f + i^k g)^*(f + i^k g))\| \\ &\leq \sum_{k=0}^3 \|\psi((f + i^k g)^*(f + i^k g))\| \leq 4(\|f\|_X + \|g\|_X)^2 , \end{aligned}$$

where for the last inequality we have used the last part of Proposition 3.2. Now replace  $f$  by  $f/\|f\|_X$  and  $g$  by  $g/\|g\|_X$  to obtain the desired inequality.  $\square$

We deduce next the application of Theorem 2.6 to the present case.

**3.4 Theorem.** *Let  $f \in B^+$ . Then  $f \in \mathcal{P}_X$  if and only if there is an  $m \in M(A)^+$  such that for every positive  $\omega \in A'$  the function  $x \mapsto \omega(f(x))$  is integrable in the ordinary sense and*

$$\int \omega(f(x))dx = \omega(m) .$$

PROOF. If  $f \in \mathcal{P}_X$  then by Proposition 1.8 the function  $x \mapsto \omega(f(x))$  is integrable and the above equation holds by the comment just before Definition 2.8.

Conversely, if  $m$  exists as in the statement of the proposition, then for any  $\lambda \in \mathcal{B}$  we have  $\lambda(x)\omega(f(x)) \leq \omega(f(x))$ . Since  $x \mapsto \omega(f(x))$  is integrable and the  $\lambda$ 's converge up to 1 pointwise, we can pass to a suitable sequence of  $\lambda$ 's to which we can apply the monotone convergence theorem to conclude that the net  $\{\omega(p_\lambda(f))\}$  converges up to  $\omega(m)$ . We are now exactly in position to apply Theorem 2.6.  $\square$

## 4. Proper Actions.

We now return to the case of an action  $\alpha$  of a group  $G$  on a  $C^*$ -algebra  $A$ . As suggested earlier, we can view  $A$  as embedded in  $C_b(G, A)$  by sending  $a \in A$  to the function  $x \mapsto \alpha_x(a)$ . We can then apply our earlier discussion to this subalgebra. However, for our later discussion of morphisms we need a slight generalization of this situation. The action  $\alpha$  extends to an action, still denoted by  $\alpha$ , on  $M(A)$ , which need not be strongly continuous. If we view  $M(A)$  as included in  $A''$  [Pe2], this action is just the restriction of that on  $A''$  used in section 1. We let  $M(A)_e$  denote the “ $\alpha$ -essential” part of  $M(A)$  for this action, that is, the  $C^*$ -subalgebra of elements on which  $\alpha$  is strongly continuous. Then we can extend  $p_\lambda$  to  $M(A)_e$  by the same formula as before.

**4.1 Definition.** We say that  $n \in (M(A)_e)^+$  is  $\alpha$ -proper if there exists an  $m \in M(A)$  such that the net  $\{p_\lambda(n)\}_{\lambda \in \mathcal{B}}$  converges strictly to  $m$ , where now

$$p_\lambda(n) = \int_G \lambda(x) \alpha_x(n) dx.$$

We denote the hereditary cone of  $\alpha$ -proper positive elements by  $\tilde{\mathcal{P}}_\alpha^+$  (notice that Lemma 2.7 applies here), and the corresponding strict  $C^*$ -weight from  $M(A)$  towards itself by  $\tilde{\psi}_\alpha$ . We let  $\mathcal{P}^+ = \tilde{\mathcal{P}}_\alpha^+ \cap A$ , a hereditary cone in  $A$ . We have the corresponding left ideals  $\tilde{\mathcal{Q}}_\alpha$  and  $\mathcal{Q}_\alpha$ , and  $*$ -subalgebra  $\tilde{\mathcal{P}}_\alpha$  and  $\mathcal{P}_\alpha$ , and we let  $\psi_\alpha$  be the restriction of  $\tilde{\psi}_\alpha$  to  $\mathcal{P}_\alpha$ .

We remark that, in contrast to [Qg, QR], our restriction to the  $\alpha$ -essential part of  $M(A)$  is required in order for  $p_\lambda$  to be defined by an ordinary integral. For example, when  $A = C_\infty(\mathbb{R})$  and  $\alpha$  is the action of  $\mathbb{R}$  by translation,  $M(A)_e$  consists of the uniformly continuous bounded functions. But there exist functions in  $C_b(\mathbb{R}) \cap L_1(\mathbb{R})$  which are not uniformly continuous. For such a function  $g$  our definition of  $p_\lambda(g)$  will not make sense as it stands. But  $x \mapsto \omega(\alpha_x(g))$  will be integrable for every finite measure  $\omega$ . One can develop a more complicated definition of  $p_\lambda$  to handle this kind of situation, but so far I have not seen a need for this.

We now give an example to show that even for the case of a proper action of  $G$  on a space  $M$ , there can be many positive  $\alpha$ -integrable elements in  $C_\infty(M)$  which are not  $\alpha$ -proper.

**4.2 Example.** Let  $M = \mathbb{R}$ , let  $G = 4\mathbb{Z}$  and let  $\alpha$  be the action of  $G$  on  $M$  by translation. This is a proper action. Let  $g_n$  be the function on  $[0, 1]$  defined by

$$g_n(t) = \begin{cases} nt & \text{for } 0 \leq t \leq 1/n \\ 1 & \text{otherwise.} \end{cases}$$

Note that the sequence  $\{g_n\}$  increases to  $\chi_{(0,1]}$ , the characteristic function of  $(0, 1]$ . Let  $h_1 = g_1$ , and for  $n \geq 2$  let  $h_n = g_n - g_{n-1}$ . Thus  $g_n = \sum_{j=1}^n h_j$ . It is easily seen that  $\|h_n\|_\infty = 1/n$ . Let  $k_n$  be the reflection of  $h_n$  about  $t = 1$ , extended to be 0 outside  $[0, 2]$ . Then  $k_n \in C_c(\mathbb{R})$ ,  $\|k_n\|_\infty = 1/n$ , and  $\sum_{j=1}^n k_j$  converges up to  $\chi_{(0,2]}$ . Let  $L_n$  be translation by  $4n$ . Set

$$f = \sum_{n=1}^{\infty} L_n k_n,$$

where we note that the convergence is uniform. Since  $\|k_n\|_\infty = 1/n$ , it follows that  $f \in C_\infty(\mathbb{R})$ . Then it is easily seen that  $f$  is  $\alpha$ -integrable. But if we identify  $M/\alpha$  with the fundamental domain  $[0, 4)$ , then it is easily seen that  $\sum_{x \in G} \alpha_x(f)$  is  $\chi_{(0,2)}$ , which is not in  $M(A)$ .

We remark that a related example appears as example 2.4 of [FMT].

We now have the corresponding version of Theorem 3.4. It relates our situation to definition 3.4 of [Qg] and section 1 of [QR].

**4.3 Theorem.** *Let  $n \in (M(A)_e)^+$ . Then  $n \in \tilde{\mathcal{P}}_\alpha^+$  if and only if there is an  $m \in M(A)^+$  such that for every positive  $\omega \in A'$ , viewed also as a linear functional on  $M(A)$ , the function  $x \mapsto \omega(\alpha_x(n))$  is integrable on  $G$  and*

$$\int \omega(\alpha_x(n)) dx = \omega(m).$$

*Proof.* Suppose that  $n \in \tilde{\mathcal{P}}_\alpha^+$  and  $\omega \in A'$ . It follows from Proposition 1.8 and the comments made just before Definition 2.8 that  $x \mapsto \omega(\alpha_x(n))$  is integrable, with integral  $\omega(\psi(m))$ .

Suppose conversely that  $x \mapsto \omega(\alpha_x(n))$  is integrable for all positive  $\omega \in A'$ , and that there is an  $m \in M(A)^+$  such that  $\int \omega(\alpha_x(n)) dx = \omega(m)$  for all  $\omega$ . Much as in the proof of Proposition 1.8, the net  $\{\omega(p_\lambda(n))\}$  converges to  $\omega(m)$  as  $\lambda$  ranges through  $\mathcal{B}$ . From Proposition 2.6 it follows that  $p_\alpha(n)$  converges to  $m$  strictly. Thus  $n \in \tilde{\mathcal{P}}_\alpha^+$  as desired.  $\square$

Exel [E1, E2] defines  $a \in A$  to be  $\alpha$ -integrable if for all  $b \in B$  the functions  $x \mapsto b\alpha_x(a)$  and  $x \mapsto \alpha_x(a)b$  are unconditionally integrable (meaning that the net  $\{\int_E b\alpha_x(a) dx\}$  for  $E$  ranging over precompact subsets of  $G$  is norm Cauchy, and similarly for  $b$  on the other side). The integrals, as  $b$  ranges over  $A$ , then define an element of  $M(A)$ . For general  $a \in A$  it is not clear to me whether this implies that  $a \in \mathcal{P}_\alpha$ . But for positive elements we have:

**4.4 Proposition.** *Let  $a \in A^+$ . Then  $a$  is  $\alpha$ -integrable in Exel's sense iff  $a \in \mathcal{P}_\alpha^+$ .*

*Proof.* Suppose that  $a$  is  $\alpha$ -integrable in Exel's sense. Exel points out (before 6.2 of [E1]) that  $a$  is then integrable in the sense used in Olesen and Pedersen discussed above (though they only consider positive elements). So if  $a \in A^+$  then we can apply Theorem 4.3 to conclude that  $a \in \mathcal{P}_\alpha^+$ .

Conversely, if  $a \in \mathcal{P}_\alpha^+$  then for every  $b \in A$  the net  $\{bp_\lambda(a) : \lambda \in \mathcal{B}\}$  is norm Cauchy by definition, and similarly for  $b$  on the other side. The subnet obtained by restricting the  $f$ 's to be characteristic functions of precompact subsets of  $G$  is clearly cofinal, so this subnet too is Cauchy. Similarly for  $b$  on the other side. Thus  $a$  is  $\alpha$ -integrable in Exel's sense.  $\square$

We remark that it follows by linearity that any element of  $\mathcal{P}_\alpha$  is  $\alpha$ -integrable in Exel's sense.

We tentatively make the following definition, which is the main one of this paper. The reason that this definition is tentative will be explained in section 6.

**4.5 Definition.** The action  $\alpha$  of  $G$  on  $A$  is *proper* if  $\mathcal{P}_\alpha$  is dense in  $A$ .

We will see later in Proposition 6.8 that every action  $\alpha$  has a canonical proper part, namely its restriction to the closure of  $\mathcal{P}_\alpha$ .

We show now that all of the examples successfully treated by the definition of [Rf7] are examples of proper actions in the sense of Definition 4.5. (See [Rf8] and [M] for further such examples in addition to those already described in [Rf7].) This already gives a substantial supply of interesting examples. The main theorem of [E1] provides yet a further class of examples, associated to  $C^*$ -algebraic bundles over locally compact Abelian groups, to which Definition 4.5 applies.

**4.6 Proposition.** *If the action  $\alpha$  of  $G$  on  $A$  is proper in the sense of definition 1.2 of [Rf7], then it is proper in the sense of Definition 4.5 above.*

*Proof.* We recall that if  $\alpha$  is proper in the sense of definition 1.2 of [Rf7], then there is a dense  $*$ -subalgebra  $A_0$  of  $A$  such that if  $a, b \in A_0$  then the function  $x \mapsto a\alpha_x(b)$  is integrable on  $G$ , and for  $a, b \in A_0$  there is an  $m \in M(A)^\alpha$  such that for every  $c \in A_0$  we have

$$\int c\alpha_x(a^*b)dx = cm.$$

(There is more to the definition, but this suffices for our present purposes.)

We claim that  $A_0 \subseteq Q_\alpha$ , so that  $A_0^2 \subseteq \mathcal{P}_\alpha$ . Since  $A_0^2$  (linear span of products) is dense in  $A$  because  $A_0$  is, it will follow that  $\alpha$  is proper in the sense of Definition 4.5.

So suppose that  $a \in A_0$ . By hypothesis there is an  $m \in M(A)^\alpha$  such that

$$\int c\alpha_x(a^*a)dx = cm$$

for every  $c \in A_0$ . For a given  $c \in A_0$  the function  $x \mapsto c\alpha_x(a^*a)$  is by assumption integrable on  $G$ , and so we can find an increasing sequence  $\{\lambda_n\}$  in  $\mathcal{B}(G)$  such that  $\{\lambda_n(x)c\alpha_x(a^*a)\}$  converges pointwise to  $c\alpha_x(a^*a)$ . By the Lebesgue dominated convergence theorem,  $c \int \lambda_n(x)\alpha_x(a^*a)dx$  converges in norm to  $cm$ . Then  $cp_{\lambda_n}(a^*a)c^*$  increases up to  $cmc^*$  in norm. Thus  $m \geq 0$  since  $A_0$  is dense. Furthermore  $cp_{\lambda_n}(a^*a)c^* \leq cmc^*$  for all  $c \in A_0$ , and it follows that  $p_{\lambda_n}(a^*a) \leq m$ . Since our sequence  $\{\lambda_n\}$  can include any given element of  $\mathcal{B}$ , it follows that  $p_\lambda(a^*a) \leq m$  for all  $\lambda \in \mathcal{B}$ , so that  $a^*a$  is  $\alpha$ -integrable.

Finally, it is easily seen that the collection of  $c$ 's in  $A$  for which  $cp_\lambda(a^*a)c^*$  converges to  $cmc^*$  is norm closed. But it contains  $A_0$ , and thus it is all of  $A$ . Hence, the net  $\{p_\lambda(a^*a)\}$  converges up to  $m$  in the weak-strict topology. But we saw in Proposition 2.4 that this implies that  $p_\lambda(a^*a)$  converges strictly to  $m$ .  $\square$

We now want to show that if  $A$  is commutative, then our definition of proper action on  $A$  captures the usual notion of proper action on a space. This is a somewhat subtle matter, as seen by examining Example 1.18. In fact, already Green's original example [G] will do — he was concerned with closely related matters. His example is the case of Example 1.18 in which  $\{L_n\}$  is the constant sequence  $L_n = 1$ . In this case  $\alpha$  is an integrable action. But for  $f$  as constructed in Example 1.18 we have

$$\int \alpha_t(f)dt = \begin{cases} 1 & \text{on } \mathcal{O}_* \\ 2 & \text{on } \mathcal{O}_n \text{ for } n \geq 1, \end{cases}$$

which is not continuous on  $M/\alpha$ . Thus  $\alpha$  is not proper as an action on  $C_\infty(M)$ , much as Green [G] showed that  $\alpha$  is not proper as an action on  $M$ . (As Green suggests there [G], the study of the transformation group  $C^*$ -algebras for actions of the kind described in Example 1.18 might be of some interest.)

**4.7 Theorem.** *Let  $\alpha$  be an action of a locally compact group  $G$  on a locally compact space  $M$ , and so on  $A = C_\infty(M)$ . Then  $\alpha$  as action on  $A$  is proper in the sense of Definition 4.5 if and only if  $\alpha$  as an action on  $M$  is proper.*

*Proof.* If  $\alpha$  on  $M$  is proper, then from the discussion at the beginning of Section 2 it follows that  $C_c(M)$  consists of  $\alpha$ -proper elements, so that  $\alpha$  on  $C_\infty(M)$  is proper. Equivalently, definition 1.2 of [Rf7] applies, so we can invoke Proposition 4.6.

Suppose, conversely, that  $\alpha$  is proper as action on  $A$ . We show that then  $\alpha$  is proper as action on  $M$ . We can assume that the  $\alpha$ -orbits in  $M$  are closed, and that the stability subgroups are compact, for otherwise  $\alpha$  on  $A$  is not even integrable, by Proposition 1.17. Since  $\mathcal{P}_\alpha$  is assumed dense, and is an ideal in this commutative case, it contains  $C_c(M)$ .

Let us show first that  $M/\alpha$  is Hausdorff. As mentioned in section 1, this does not follow from integrability of  $\alpha$ . Let  $m, n \in M$  and suppose they are in different  $\alpha$ -orbits. Since the orbit  $\alpha_G(n)$  is closed, we can find  $f \in C_c(M)^+$  such that  $f(m) > 0$  while  $f(\alpha_G(n)) = 0$ . Since  $f \in \mathcal{P}_\alpha$ ,  $F = \int \alpha_x(f)dx$  exists and is continuous. Clearly  $F(m) > 0$  while  $F(n) = 0$ . Thus  $F$  is a continuous function on  $M/\alpha$  which separates  $m$  and  $n$ . Since  $m$  and  $n$  are arbitrary, it follows that  $M/\alpha$  is Hausdorff. So we assume from now on that  $M/\alpha$  is Hausdorff.

Suppose now that  $\alpha$  on  $M$  is not proper. It is easily seen from the definition that there is then a compact subset,  $K$ , of  $M$  such that  $\{x \in G : \alpha_x(K) \cap K = \emptyset\}$  is not precompact. Thus we can choose a net  $\{k_\mu\}$  of elements of  $K$  and a net  $\{x_\mu\}$  of elements of  $G$  such that  $\alpha_{x_\mu}(k_\mu) \in K$  for each  $\mu$ , but the net  $\{x_\mu\}$  is not precompact. By the compactness of  $K \times K$  we can find a subnet  $\{(x_\nu, k_\nu)\}$  of the net  $\{(x_\mu, k_\mu)\}$  such that  $k_\nu \rightarrow k_0$  and  $\alpha_{x_\nu}(k_\nu) \rightarrow k'_0$  for points  $k_0$  and  $k'_0$  of  $K$ . Since  $M/\alpha$  is Hausdorff, it follows that  $k_0$  and  $k'_0$  are in the same  $\alpha$ -orbit, so there is a  $y \in G$  with  $k'_0 = \alpha_y(k_0)$ . If we replace each  $x_\nu$  by  $(y^{-1}x_\nu)^{-1}$ , we find that  $\alpha_{x_\nu^{-1}}(k_\nu) \rightarrow k_0$ .

Choose  $f \in C_c(M)^+$  such that  $\int f(\alpha_{y^{-1}}(k_0))dy = 1$ . Since the orbit of  $k_0$  is closed, it meets the support of  $f$  in a compact set. Since the stability subgroup of  $k_0$  is compact, we can find a compact subset  $C$  of  $G$  such that

$$\int_C f(\alpha_{y^{-1}}(k_0))dy = \int_G f(\alpha_{y^{-1}}(k_0))dy = 1.$$

Let  $\chi$  denote the characteristic function of  $C$ , so  $\chi \in \mathcal{B}$ . Then

$$(p_\chi(f))(k_0) = \int_C f(\alpha_{y^{-1}}(k_0))dy = 1.$$

Now  $p_\chi(f)$  is continuous, and so  $(p_\chi(f))(k_\nu) \rightarrow 1$  and  $(p_\chi(f))(\alpha_{x_\nu^{-1}}(k_\nu)) \rightarrow 1$ . But

$$(p_\chi(f))(\alpha_{x_\nu^{-1}}(k_\nu)) = \int_C f(\alpha_{(x_\nu y)^{-1}}(k_\nu))dy = \int_{x_\nu C} f(\alpha_{y^{-1}}(k_\nu))dy.$$

Since the net  $\{x_\nu\}$  is not precompact, it is not eventually contained in  $CC^{-1}$ . So we can find a subnet, which we still denote by  $\{x_\nu\}$ , such that  $x_\nu \notin CC^{-1}$  for all  $\nu$ . Then  $C$  and  $x_\nu C$  are disjoint for each  $\nu$ , so

$$\int_G f(\alpha_{y^{-1}}(k_\nu)) \geq \int_C f(\alpha_{y^{-1}}(k_\nu)) dy + \int_{x_\nu C} f(\alpha_{y^{-1}}(k_\nu)) dy,$$

which converges to  $1 + 1 = 2$ . Thus

$$\liminf \int f(\alpha_{y^{-1}}(k_\nu)) dy \geq 2.$$

Since  $\int f(\alpha_{y^{-1}}(k_0)) dy = 1$ , we see that  $\int \alpha_y(f) dy$  is not continuous on  $M$ , so  $\alpha$  as action on  $A$  is not proper, a contradiction.  $\square$

We conclude this section by showing that the  $C^*$ -weights in the present context have slightly better continuity properties than we encountered earlier. We first need:

**4.8 Proposition.** *Let  $\alpha$  be an action of  $G$  on  $A$ , and let  $\lambda \in \mathcal{B}$ . For any bounded approximate identity  $\{e_\nu\}$  for  $A$ , the net  $\{p_\lambda(e_\nu)\}$  converges strictly in  $M(A)$  to  $(\int \lambda(x) dx)1$ .*

*Proof.* We first remark that if  $h$  is a continuous function from a compact space  $K$  to  $A$ , then for any  $\varepsilon > 0$  there is a  $\nu_0$  such that  $\|h(x) - e_\nu h(x)\| < \varepsilon$  for all  $x \in K$  and all  $\nu > \nu_0$ . The same is true for  $e_\nu$  on the right of  $h(x)$ . This follows by using the compactness of  $K$  to approximate  $h$  by a finite sum  $\sum \varphi_j h(x_j)$  where  $\{\varphi_j\}$  is a suitable partition of unity on  $K$ . Now let  $K$  denote the (compact) support of  $f$ . Let  $\varepsilon > 0$  and  $c \in A$  be given. Then

$$\begin{aligned} cp_\lambda(e_\nu) - c(\int \lambda(x) dx) &= \int \lambda(x)(c\alpha_x(e_\nu) - c) dx \\ &= \int \lambda(x)\alpha_x(\alpha_{x^{-1}}(c)e_\nu - \alpha_{x^{-1}}(c)) dx. \end{aligned}$$

When we apply the above remark to the function  $x \mapsto \alpha_{x^{-1}}(c)$ , we see that we can find  $\nu_0$  such that

$$\|cp_\lambda(e_\nu) - c(\int \lambda(x) dx)\| < \varepsilon \text{ for } \nu > \nu_0.$$

Taking adjoints, we obtain the corresponding result for  $c$  on the other side.  $\square$

**4.9 Definition.** A completely positive map  $p$  from  $B$  to  $A$  is said to be *non-degenerate* if for some bounded approximate identity  $\{e_\lambda\}$  for  $B$  the net  $\{p(e_\lambda)\}$  converges strictly to  $r1 \in M(A)$  for some  $r \in \mathbb{R}^+$ .

This is just the definition at the top of page 49 of [La] except that we do not require that  $\|p\| = 1$ . Because we here require that  $p$  be *completely* positive, we can apply some of the results in [La]. In particular, by Lemma 5.3 of [La] we will have  $r = \|p\|$ .

Notice now that Proposition 4.8 states that each  $p_\lambda$  is non-degenerate, for  $\lambda \in \mathcal{B}$ .

**4.10 Definition.** We will say that a strict  $C^*$ -weight  $\psi$  from  $B$  toward  $A$  is *non-degenerate* if there is an increasing net  $P = \{p_\lambda\}$  of completely positive maps from  $B$  to  $A$  as in Definition 2.8 such that eventually each  $p_\lambda$  is non-degenerate.

According to Corollary 5.7 of [La], if  $p$  is non-degenerate, then  $p$  extends uniquely to a completely positive map  $\bar{p}$  from  $M(B)$  to  $M(A)$  such that  $\bar{p}(1_{M(B)}) = \|p\|1_{M(A)}$ , and  $\bar{p}$  is strictly continuous on bounded subsets of  $M(B)$ . This makes possible the following strengthening of the lower semi-continuity property stated in Proposition 2.9, when  $\psi$  is non-degenerate.

**4.11 Proposition.** (Compare with 3.5 of [Ku].) *Let  $\psi$  be a strict  $C^*$ -weight from  $B$  toward  $A$ , and assume that  $\psi$  is (completely positive and) non-degenerate. Let  $b \in \mathcal{P}_P^+$ , and let  $\{b_\mu\}$  be a net in  $B^+$  which converges strictly to  $b$  and is such that  $b_\mu \leq b$  for each  $\mu$ . Then the net  $\{\psi(b_\mu)\}$  converges strictly to  $\psi(b)$ .*

*Proof.* The proof is the same as that for Proposition 2.9 except that now we use the strict continuity of  $p_\lambda$  in order to choose  $\mu_0$  such that for  $\mu \geq \mu_0$

$$\|ap_\lambda(b)a^* - ap_\lambda(b_\mu)a^*\| < \varepsilon/2.$$

□

## 5. Functoriality, and $C^*$ -algebras proper over a space.

In considering functoriality it is useful for us to treat “morphisms” [La, Wr]. Let  $A$  and  $B$  be  $C^*$ -algebras. A morphism from  $B$  to  $A$  is a homomorphism  $\theta$  from  $B$  to  $M(A)$  which is non-degenerate in the sense that  $\theta(B)A$  is dense in  $A$ . Then  $\theta$  extends uniquely to a homomorphism,  $\bar{\theta}$ , from  $M(B)$  to  $M(A)$ , which is strictly continuous on bounded sets [La]. If  $\alpha$  and  $\beta$  are actions of  $G$  on  $A$  and  $B$ , then we say that  $\theta$  is equivariant if

$$\theta(\beta_x(b)) = \alpha_x(\theta(b))$$

for all  $x \in G$  and  $b \in B$ , where here  $\alpha$  has been extended to  $M(A)$ . The extension of  $\theta$  to  $M(B)$  is then seen to be equivariant in the usual sense. The following proposition is basically proposition 1.4 of [QR] once Theorem 2.6 above is taken into account.

**5.1 Proposition.** *With  $\alpha$ ,  $\beta$  and  $\theta$  as above, we have  $\bar{\theta}(\tilde{\mathcal{P}}_\beta) \subseteq \tilde{\mathcal{P}}_\alpha$ , and*

$$\tilde{\psi}_\alpha(\bar{\theta}(n)) = \bar{\theta}(\tilde{\psi}_\beta(n))$$

for all  $n \in \tilde{\mathcal{P}}_\beta$ .

*Proof.* It is easily seen that  $\bar{\theta}(M(B)_e) \subseteq M(A)_e$ . Let  $n \in \tilde{\mathcal{P}}_\beta$ . By definition the net  $\{p_\lambda^\beta(n)\}$  is bounded and converges to  $\tilde{\psi}_\beta(n)$  strictly. Thus  $\{\bar{\theta}(p_\lambda^\beta(n))\}$  converges strictly to  $\bar{\theta}(\tilde{\psi}_\beta(n))$ . But  $\bar{\theta}(p_\lambda^\beta(n)) = p_\lambda^\alpha(\bar{\theta}(n))$ . □

The next lemma should be compared carefully with the definition of hereditary (non-closed) subalgebras in VII.4.1 of [FD].

**5.2 Lemma.** *Let  $\mathcal{H}$  be any hereditary  $*$ -subalgebra, possibly not closed, of a  $C^*$ -algebra  $C$ . Then  $\mathcal{H}C\mathcal{H} \subseteq \mathcal{H}$ . Let  $D$  be the closure  $(\mathcal{H}C\mathcal{H})^-$ , where  $\mathcal{H}C\mathcal{H}$  means linear span. Then  $\mathcal{H}C\mathcal{H} \cap C^+$  is dense in  $D^+$ . Furthermore, the closure,  $\bar{\mathcal{H}}$ , of  $\mathcal{H}$  in  $C$  is a hereditary  $C^*$ -subalgebra of  $C$ .*

*Proof.* Here, in contrast to [FD], by “hereditary” we mean that  $\mathcal{H}$  is the linear span of its positive part  $\mathcal{H}^+$ , and that  $\mathcal{H}^+$  is a hereditary cone in  $C$  in the sense we used earlier. Let  $c \in C^+$  and  $h \in \mathcal{H}$ . Then  $h^*ch \leq \|c\|h^*h$ , so that  $h^*ch \in \mathcal{H}$ . By linearity it follows that this is true for any  $c \in C$ . By polarization it then follows that if  $h_1, h_2 \in \mathcal{H}$  and  $c \in C$ , then  $h_1ch_2 \in \mathcal{H}$ .

Now suppose that  $d \in D^+$ . By considering an approximate identity for  $D$ , and approximating its elements by elements of  $\mathcal{H}C\mathcal{H}$ , we can approximate  $d$  by elements of form  $b^*db$  where  $b \in \mathcal{H}C\mathcal{H}$ . But then  $b^*db \in (\mathcal{H}C\mathcal{H}) \cap C^+$ .

Finally, it is clear that  $\bar{\mathcal{H}}C\bar{\mathcal{H}} \subseteq \bar{\mathcal{H}}$ . But as indicated in VII.4.1 of [FD] this does imply that  $\bar{\mathcal{H}}$  is hereditary in our sense, since  $\bar{\mathcal{H}}$  is closed.  $\square$

It is easily seen that if there is an equivariant map from a  $G$ -space  $Y$  to a  $G$ -space  $Z$  and if the action on  $Z$  is proper, then the action on  $Y$  must be proper. We have the following generalization to the non-commutative case:

**5.3 Theorem.** *Let  $\alpha$  and  $\beta$  be actions of  $G$  on  $C^*$ -algebras  $A$  and  $B$ , and suppose that  $\beta$  is proper. If there is an equivariant morphism from  $B$  to  $A$ , then  $\alpha$  is proper.*

*Proof.* Let  $\theta$  be an equivariant morphism from  $B$  to  $A$ . Since  $\theta$  is non-degenerate,  $\theta(B)A\theta(B)$  is dense in  $A$ . Since  $\beta$  is proper,  $\mathcal{P}_\beta$  is dense in  $B$ , and so  $\theta(\mathcal{P}_\beta)A\theta(\mathcal{P}_\beta)$  is dense in  $A$ . But  $\theta(\mathcal{P}_\beta) \subseteq \tilde{\mathcal{P}}_\alpha$  by Proposition 5.1. Thus  $\tilde{\mathcal{P}}_\alpha A \tilde{\mathcal{P}}_\alpha$  is dense in  $A$ . But  $\tilde{\mathcal{P}}_\alpha$  is a hereditary  $*$ -subalgebra in  $M(A)$  by Key Lemma 2.7. Thus  $\tilde{\mathcal{P}}_\alpha A \tilde{\mathcal{P}}_\alpha \subseteq \tilde{\mathcal{P}}_\alpha$  by Lemma 5.2. Since  $\mathcal{P}_\alpha = \tilde{\mathcal{P}}_\alpha \cap A$ , it follows that  $\mathcal{P}_\alpha$  is dense in  $A$ .  $\square$

**5.4 Corollary.** *Let  $\alpha$  be a proper action of  $G$  on a  $C^*$ -algebra  $A$ , and let  $I$  be an  $\alpha$ -invariant ideal in  $A$ , so that  $\alpha$  drops to an action,  $\bar{\alpha}$ , on  $A/I$ . Then  $\bar{\alpha}$  is proper.*

**5.5 Proposition.** *For  $\alpha$ ,  $A$  and  $I$  as just above, the action defined by  $\alpha$  on  $I$  is proper.*

*Proof.* Since  $\mathcal{P}_\alpha$  is dense in  $A$ , the linear span  $\mathcal{P}_\alpha I \mathcal{P}_\alpha$  must be dense in  $I$ . But it is contained in  $\mathcal{P}_\alpha$  by Lemma 5.2. Thus  $\mathcal{P}_\alpha \cap I$  is dense in  $I$ . In fact, from Lemma 5.2 we see that  $\mathcal{P}_\alpha \cap I^+$  is dense in  $I^+$ . Each element of  $M(A)$  determines an element of  $M(I)$  in the evident way. Let  $c \in \mathcal{P}_\alpha \cap I^+$ , and let  $\psi_\alpha(c)$  also denote the corresponding element of  $M(I)$ . It is easily seen that  $\{p_\lambda(c)\}$  converges strictly to  $\psi_\alpha(c)$  in  $M(I)$ .  $\square$

An increasingly important way of getting aspects of properness to bear on an action of a group on a  $C^*$ -algebra is to have an equivariant morphism from a commutative  $C^*$ -algebra with proper action, whose image is central. This technique seems to have been first introduced by Kasparov, in section 3 of [Ks]. For more recent occurrences see [GHT] and the references therein. Such a morphism is a special case of the situation of Theorem 5.3, so that we immediately obtain:

**5.6 Corollary.** *Let  $\alpha$  be an action of  $G$  on a  $C^*$ -algebra  $A$ . Let  $\beta$  be a proper action of  $G$  on a locally compact space  $Z$ , and so on  $C_\infty(Z)$ . If there is an equivariant morphism from  $C_\infty(Z)$  to  $A$  whose image is contained in the center of  $M(A)$ , then  $\alpha$  is proper.*

The deficiency of this corollary is that, as we discuss in the next section, we have not seen how to establish strong Morita equivalence between the generalized fixed-point algebra and an ideal in the crossed product in the general situation of our present definition of properness. However, we now show that, even in the absence of the requirement that the image of the morphism be central, the situation of Corollary 5.6 falls within the purview of definition 1.2 of [Rf7], where we were able to establish this Morita equivalence. The fact that centrality is not needed seems to be a new observation. Our proof can be viewed as a variation of the proof of theorem 3.13 of [Ks], with some ingredients also from [Qg]. We will not include discussion of the fact that if the action on  $Z$  is also free, then the strong Morita equivalence is with the whole crossed product, but see the discussion of “saturation” in [Rf7].

**5.7 Theorem.** *Let  $\alpha$  be an action of  $G$  on a  $C^*$ -algebra  $A$ . Let  $\beta$  be a proper action of  $G$  on a locally compact space  $Z$ , and so on  $C_\infty(Z)$ , and let  $\theta$  be an equivariant morphism from  $C_\infty(Z)$  to  $A$ . Let  $A_0$  denote the linear span of  $(\theta(C_c(Z))A(\theta(C_c(Z)))$ , which is a dense  $*$ -subalgebra of  $A$ . Then  $A_0$  satisfies the conditions of definition 1.2 of [Rf7], so that  $\alpha$  is proper in the sense of that definition. Thus the generalized fixed-point algebra (in the sense of [Rf7]) is strongly Morita equivalent to an ideal in the reduced crossed product algebra.*

*Proof.* For notational simplicity we sometimes omit  $\theta$  below, and confuse  $\beta$  with  $\alpha$ . Let  $a, b \in A$  and  $f, g \in C_c(Z)$ , and consider the function

$$x \mapsto (af)\alpha_x(gb) = a(f\beta_x(g))\alpha_x(b).$$

Since  $\beta$  is proper, this function has compact support. From this it is easily seen that if  $a, b \in A_0$ , then the function  $x \mapsto a\alpha_x(b)$  has compact support, and so is in  $L^1(G, A)$ , as will be its product with  $\Delta^{-1/2}$ . This says exactly that condition 1 of definition 1.2 of [Rf7] is satisfied.

We turn now to condition 2. By essentially the same argument as in the proof of Theorem 5.3, using the fact that  $C_c(Z) \subseteq \mathcal{P}_\beta$ , we find that  $A_0 \subseteq \mathcal{P}_\alpha$ . For the element  $\langle a, b \rangle_D$  of  $M(A)^\alpha$  which is required by condition 2 for any  $a, b \in A_0$  we take  $\psi_\alpha(a^*b)$ . (See Proposition 1.12 for the  $\alpha$ -invariance.) Then for  $c \in A_0$  we have

$$c\langle a, b \rangle_D = \lim \int \lambda(x) c\alpha_x(a^*b) dx.$$

But as seen above,  $x \mapsto c\alpha_x(a^*b)$  has compact support, and so the net of integrals is eventually constant, and has limit

$$\int c\alpha_x(a^*b) dx,$$

as required by condition 2. Now condition 2 also requires that  $c\langle a, b \rangle_D$  be again in  $A_0$  for  $a, b, c \in A_0$ , that is, that  $\langle a, b \rangle_D \in M(A_0)^\alpha$  in the notation of [Rf7]. It is easily seen that it suffices to show that if  $a, b \in A$  and if  $f_1, f_2, g_1, g_2 \in C_c(Z)$ , then  $f_1 a f_2 \psi_\alpha(g_1 b g_2) \in A_0$ . But by the argument from the proof of condition 1 above we see that this element is given by

$$f_1 \int a(f_2 \alpha_x(g_1)) \alpha_x(b) \alpha_x(g_2) dx.$$

Let  $K$  denote the support of the integrand, which is compact. Let  $S$  denote the support of  $g_2$ , and let  $L = \alpha_K(S)$ . Choose  $h \in C_c(Z)$  such that  $h \equiv 1$  on  $L$ . Then for  $x \in K$  we have  $\alpha_x(g_2) = \alpha_x(g_2)h$ . Consequently the above expression can be rewritten as

$$f_1\left(\int a(f_2\alpha_x(g_1))\alpha_x(b)\alpha_x(g_2)dx\right)h,$$

which is manifestly in  $A_0$ . □

It is not clear to me how Thomsen's definition of a K-proper action, given in 9.1 of [T], relates to our present considerations, though it has some relation to [GHT].

## 6. Strong Morita equivalence.

In this section we will discuss what one might take as the “generalized fixed-point algebra” for a proper action. Our guiding principle will be that this generalized fixed-point algebra should be strongly Morita equivalent [Rf2, Rf4, Rf5] to at least an ideal in the crossed product algebra, much as happens in [Rf7]. (In the case of commutative  $A = C_\infty(X)$  we know [Rf7] that it will be strongly Morita equivalent to the entire reduced crossed product algebra exactly if the action on  $X$  is free.) The outcome of our discussion will be far from satisfactory. In particular, Exel [E2] has shown that the candidate for “generalized fixed-point algebra” which I had suggested in the first version of this paper is often too big. (See our discussion following Theorem 8.5.) In fact, Exel [E2] has shown that the situation is fairly subtle, and the question of how best to define the generalized fixed-point algebra remains unresolved.

We let  $M(A)^\alpha$  denote the subalgebra of elements in  $M(A)$  which are fixed by  $\alpha$ . From Proposition 1.12 we immediately obtain:

**6.1 Proposition.** *The range of  $\psi_\alpha$  is contained in  $M(A)^\alpha$ .*

By viewing elements of  $M(A)^\alpha$  as constant functions on  $G$  and applying Proposition 3.1 we obtain:

**6.2 Proposition.** *Let  $a \in \mathcal{P}_\alpha$  and let  $m \in M(A)^\alpha$ . Then  $ma$  and  $am$  are in  $\mathcal{P}_\alpha$ , and*

$$\psi_\alpha(ma) = m\psi_\alpha(a), \quad \psi_\alpha(am) = \psi_\alpha(a)m.$$

*If  $a \in \mathcal{Q}_\alpha$  then  $am \in \mathcal{Q}_\alpha$ .*

We note that this proposition implies that the range of  $\psi$  is an ideal in  $M(A)^\alpha$  (and clearly a  $*$ -ideal).

It is clear from Proposition 3.1 that  $\mathcal{P}_\alpha$  and  $\mathcal{Q}_\alpha$  are right  $M(A)^\alpha$ -modules. Then  $\psi_\alpha$  is almost a generalized conditional expectation from  $\mathcal{P}_\alpha$ , as defined in definition 4.12 of [Rf2]. The only property which is not clear is the  $\psi$ -density of  $\mathcal{P}_\alpha^2$  in  $\mathcal{P}_\alpha$  as defined in property 5 of definition 4.12 of [Rf2]. I do not know how often it holds. (The relative boundedness of property 4 of the definition follows from the fact that for  $b \in \mathcal{P}_\alpha$  the map  $a \mapsto \psi_\alpha(b^*ab)$  is defined on all of  $A$  and is positive, and so is bounded.)

We remark that the KSGNS construction of [Ku] can, of course, be carried out in our special case. Because of the above conditional expectation property of  $\psi_\alpha$ , the KSGNS construction for  $\psi_\alpha$  is in this case essentially the “induction in stages” construction of theorem 6.9 of [Rf2], applied to  $\mathcal{Q}_\alpha$  as left- $A$  right-Hilbert- $M(A)^\alpha$ -module using  $\psi_\alpha$ , and  $A$  as left- $M(A)^\alpha$  right-Hilbert- $A$ -module in the canonical way. By 3.7 of [Ku] the construction gives a non-degenerate representation of  $A$ . The proof of non-degeneracy basically uses Proposition 2.12 above.

It is at first sight not entirely clear what one should take as the “generalized fixed-point algebra”. Our guiding principle will be our desire, just expressed above, that it be strongly Morita equivalent to at least an ideal in the crossed product algebra. One possibility is to take the generalized fixed-point algebra to be the closure of the range of  $\psi_\alpha$ . We now give an example to show that already when  $A$  is commutative this does not accord with our guiding principle.

**6.3 Example.** Let  $G$ ,  $A$  and  $\alpha$  be as in Example 2.10. Then it is easily seen that  $A \times_\alpha G$  is isomorphic to the  $C^*$ -direct-sum of a countable number of compact operator algebras. Its primitive ideal space is thus a countable discrete set, and so it cannot be strongly Morita equivalent to a unital  $C^*$ -algebra, since strongly Morita  $C^*$ -algebras have homeomorphic primitive ideal spaces (corollary 3.3 of [Rf3]). But let  $g$  be as in Example 2.10. It is seen there that  $\psi(g) = 1$ . So the closure of the range of  $\psi$  is a unital  $C^*$ -algebra, and thus cannot be strongly Morita equivalent to  $A \times_\alpha G$ . Of course the difficulty is that in this case we want the generalized fixed-point algebra to be contained in  $C_\infty(N)$ .

To try to remedy the situation we consider a slightly subtler definition. Since  $P_\alpha = \mathcal{Q}_\alpha^* \mathcal{Q}_\alpha$ , we can define an  $M(A)^\alpha$ -valued inner-product  $\langle \cdot, \cdot \rangle_D$ , on  $\mathcal{Q}_\alpha$  by

$$\langle a, b \rangle_D = \psi(a^* b) .$$

By Proposition 3.1 this behaves correctly for the right action of  $M(A)^\alpha$ . But since  $P_\alpha = \mathcal{Q}_\alpha^* \mathcal{Q}_\alpha$ , the span of the range of this inner-product is just the range of  $\psi_\alpha$ , and so by the above example this span will not be appropriate as the generalized fixed-point algebra. So instead, we consider the restriction of this inner-product to  $P_\alpha \subseteq \mathcal{Q}_\alpha$ . Exel’s example [E2] shows that this is in general still too big. (See the discussion after Theorem 8.5 below.) But we examine it here. That is, we set:

**6.4 Definition.** Let  $\alpha$  be an action of  $G$  on  $A$ . The *big generalized fixed-point algebra* of  $\alpha$  is the norm closure of the linear span of the elements of  $M(A)^\alpha$  of the form

$$\langle a, b \rangle_D = \psi_\alpha(a^* b)$$

for  $a, b \in P_\alpha$ . We will denote it by  $D_\alpha$ .

This accords with the definition given in [Rf7], as well as with definition 1.5 of [QR]. It is clear from Proposition 6.2 that  $D_\alpha$  is an ideal in  $M(A)^\alpha$ . We remark that just for the purpose of stating this definition we do not need to assume that  $\alpha$  is proper.

We now proceed to show that at least when  $A$  is commutative this definition provides a generalized fixed-point algebra where we want it.

**6.5 Proposition.** *Let  $\alpha$  be a proper action of  $G$  on a commutative  $C^*$ -algebra  $A = C_\infty(M)$ . Let  $M/\alpha$  be the orbit space, with its quotient locally compact Hausdorff topology. For  $f, g \in \mathcal{P}_\alpha$  we have*

$$\psi_\alpha(\bar{f}g) \in C_\infty(M/\alpha).$$

*Proof.* Of course  $\psi_\alpha(\bar{f}g) \in C_b(X/\alpha)$ . The only issue is the vanishing at infinity. Let  $\varepsilon \geq 0$  be given. We can find compact  $K \subseteq M$  such that  $|f(m)| \leq \varepsilon$  for  $m \notin K$ . The image,  $\dot{K}$ , of  $K$  in  $M/\alpha$  is compact. Let  $m \in M$  with  $\dot{m} \notin \dot{K}$ . View evaluation at  $m$  as a state of  $A$ , so that we can apply Theorem 3.4. Then  $x \mapsto (\bar{f}g)(\alpha_x^{-1}(m))$  is integrable on  $G$ , and

$$|(\psi(\bar{f}g))(m)| \leq \int_G |\bar{f}(\alpha_x^{-1}(m))g(\alpha_x^{-1}(m))| dx \leq \varepsilon \int_G |g(\alpha_x^{-1}(m))| dx \leq \varepsilon \|\psi_\alpha(g)\|_\infty.$$

That is,  $|\langle f, g \rangle_D(\dot{m})| \leq \varepsilon \|\psi_\alpha(g)\|_\infty$  for  $\dot{m} \notin \dot{K}$ , as desired.  $\square$

We now turn to the question of strong Morita equivalence. It is natural in view of [Rf7] to take  $\mathcal{P}_\alpha$  (suitably completed) as the equivalence bimodule. We know that it is a right  $D_\alpha$ -module. We restrict to  $\mathcal{P}_\alpha$  the inner-product defined above on  $\mathcal{Q}_\alpha$ . It then has values in  $D_\alpha$  by definition. We consider the corresponding norm  $\|a\|_\alpha = \|\langle a, a \rangle_D\|^\frac{1}{2}$ . Then the completion,  $\bar{\mathcal{P}}_\alpha$ , for this norm is a right Hilbert  $D_\alpha$ -module [La], which is *full* in the sense that the span of the range of the inner-product is dense in  $D_\alpha$  (because we defined  $D_\alpha$  that way).

Thus we have the corresponding algebra,  $B(\bar{\mathcal{P}}_\alpha)$ , of bounded (adjointable) operators on  $\bar{\mathcal{P}}_\alpha$ , and its ideal  $E$  of “compact” operators on  $\bar{\mathcal{P}}_\alpha$  [Rf2, La] generated by the “rank-one” operators  $\langle a, b \rangle_E$  defined by

$$\langle a, b \rangle_E c = a \langle b, c \rangle_D .$$

Always  $E$  is strongly Morita equivalent to  $D_\alpha$  [Rf2]. What we need to do is to relate  $E$  to the (reduced) crossed product algebra  $A \rtimes_\alpha^r G$ . So we examine the extent to which  $A \rtimes_\alpha^r G$  acts on  $\mathcal{P}_\alpha$ . We begin by considering the action of  $G$ .

For each  $x \in G$  we define an operator,  $U_x$ , on  $\mathcal{P}_\alpha$  by the same formula as in Notation 1.15. That this operator carries  $\mathcal{P}_\alpha$  into itself follows from the following more general fact:

**6.6 Proposition.** *Let  $\mu$  be a finite measure of compact support on  $G$ . For any  $a \in \mathcal{P}_\alpha$  define  $U_\mu a$  by*

$$U_\mu a = \int_G U_x a \, d\mu(x) ,$$

*in terms of the  $C^*$ -norm of  $A$ . Then  $U_\mu a \in \mathcal{P}_\alpha$ , and*

$$\psi_\alpha(U_\mu a) = \left( \int \Delta(y)^{-\frac{1}{2}} d\mu(y) \right) \psi(a) .$$

*Proof.* It suffices to prove this for the case in which  $\mu$  is positive and  $a \in \mathcal{P}_\alpha^+$ . Now for  $\lambda \in \mathcal{B}$  we have

$$\begin{aligned} p_\lambda(U_\mu a) &= \int \lambda(x) \alpha_x \left( \int \Delta(y)^{\frac{1}{2}} \alpha_y(a) d\mu(y) \right) dx \\ &= \int \Delta(y)^{\frac{1}{2}} \int \lambda(x) \alpha_{xy}(a) dx d\mu(y) = \int \Delta(y)^{\frac{1}{2}} \int \lambda(xy^{-1}) \alpha_x(a) \Delta(y^{-1}) dx d\mu(y) \\ &= \int \left( \int \lambda(xy^{-1}) \Delta(y)^{-\frac{1}{2}} d\mu(y) \right) \alpha_x(a) dx. \end{aligned}$$

Denote the inner integral by  $\lambda * \mu$ . It is in  $C_c^+(G)$ , and we can rewrite the above as  $p_\lambda(U_\mu a) = p_{(\lambda * \mu)}(a)$ . If we scale  $\mu$  so that  $\Delta(y)^{-\frac{1}{2}} d\mu(y)$  is a probability measure, then  $\lambda * \mu \in \mathcal{B}$ . Furthermore, the collection of such  $\lambda * \mu$ 's is cofinal in  $\mathcal{B}$ , since given  $\lambda_1 \in \mathcal{B}$  we can choose  $\lambda$  such that it has value 1 on  $(\text{support}(\lambda_1))(\text{support}(\mu))^{-1}$ , so that  $\lambda * \mu$  has value 1 on  $\text{support}(\lambda_1)$ . Consequently  $p_\lambda(U_\mu a)$  must converge strictly to  $\psi(a)$ . Then in view of how we scaled  $\mu$ , we obtain the desired conclusion.  $\square$

Note that if  $\mu$  does not have compact support,  $\int \Delta(y)^{-\frac{1}{2}} d\mu(y)$  may not be finite.

**6.7 Corollary.** *The action  $\alpha$  carries  $\mathcal{P}_\alpha$  into itself.*

Let me remark that I do not know whether  $U_\mu$  carries  $\mathcal{Q}_\alpha$  into itself in general.

We can now clarify a remark made after Definition 4.5.

**6.8 Proposition.** *Let  $\bar{\mathcal{P}}_\alpha$  denote the norm closure of  $\mathcal{P}_\alpha$  in  $A$ . Then  $\bar{\mathcal{P}}_\alpha$  is a hereditary  $C^*$ -subalgebra of  $A$  which is carried into itself by  $\alpha$ , and on which the action  $\alpha$  is proper.*

*Proof.* Denote  $\bar{\mathcal{P}}_\alpha$  by  $B$ . It is clear that  $B$  is a  $C^*$ -subalgebra of  $A$ , which from Corollary 6.7 is carried into itself by  $\alpha$ . Since  $\mathcal{P}_\alpha$  is hereditary, it follows from Lemma 5.2 that  $B$  is hereditary.

We now show that the action  $\alpha$  on  $B$  is proper. For clarity of argument we denote this restricted action by  $\beta$ . Let  $a \in \mathcal{P}_\alpha^+$ . It suffices to show that then  $a \in \mathcal{P}_\beta^+$ . Since  $a \in \mathcal{P}_\alpha^+$ , there is an  $m \in M(A)^+$  such that the net  $\{p_\lambda(a)\}$  converges strictly to  $m$ . It is easily seen that  $p_\lambda(a) \in B$  for each  $\lambda$ , since  $a \in B$ . For  $c \in B$  the net  $\{cp_\lambda(a)\}$  is in  $B$  and converges in norm to  $cm$ . Thus  $cm \in B$ . Similarly  $mc \in B$ . That is,  $m$  normalizes  $B$ , and so determines an element, say  $n$ , of  $M(B)$ . The above steps then show that  $\{p_\lambda(a)\}$  converges strictly for  $B$  to  $n$ . Thus  $a \in \mathcal{P}_\beta^+$  as desired.  $\square$

We remark that the above proposition does not adequately capture the notion of the wandering subset of an action on an ordinary space. For example, let  $M$  be the one-point compactification of  $\mathbb{R}$ , with action  $\alpha$  of  $G = \mathbb{R}$  by translation, leaving the point at infinity fixed. The wandering subset is  $\mathbb{R}$ , on which the action is proper. But if we set  $A = C(M)$  with corresponding action  $\alpha$ , then it is easily checked that  $\mathcal{P}_\alpha = \{0\}$ .

Since  $\psi_\alpha$  is the restriction to  $\mathcal{P}_\alpha$  of the  $\phi_\alpha$  of section 1, it follows as in Notation 1.15 that  $U_x$  is “unitary” for the  $D_\alpha$ -valued inner product on  $\bar{\mathcal{P}}_\alpha$ .

However, we also need the representation  $x \mapsto U_x$  of  $G$  on  $\bar{\mathcal{P}}_\alpha$  to be strongly continuous for the norm  $\|\cdot\|_\alpha$  on  $\bar{\mathcal{P}}_\alpha$ . I have not succeeded in showing that this holds in general, though it can be shown to hold in many examples. This kind of question is known to be

difficult even in the case of ordinary weights (in contrast to traces). See lemma 3.1 of [QV] for a fairly restrictive hypotheses, “regularity” (also discussed in [Ku]), under which one can prove this strong continuity for weights.

It is natural to expect that  $U$  is strongly continuous on vectors of the form  $U_g a$  where  $a \in \mathcal{P}_\alpha$  and  $g \in C_c(G)$  and where we view  $g$  as the measure  $\mu = g(x)dx$ . We now show that this is the case. But this then reduces our question to:

**6.9 Question..** With notation as above, is the linear span of the elements of  $\mathcal{P}_\alpha$  of the form  $U_g a$  for  $a \in \mathcal{P}_\alpha$  and  $g \in C_c(G)$ , dense in  $\mathcal{P}_\alpha$  for the norm from the  $D_\alpha$ -valued inner product?

**6.10 Lemma.** *Let  $\mu$  be any finite complex measure of compact support on  $G$  and let  $a, b \in \mathcal{P}_\alpha$ . Then*

$$\|\langle b, U_\mu a \rangle_D\| \leq 4\|\psi_\alpha(b^*b)\| \|\psi_\alpha(a^*a)\| \|\mu\|_1 ,$$

where  $\|\mu\|_1$  denotes the total variation norm of  $\mu$ .

*Proof.* Let  $\lambda \in \mathcal{B}$ . Then by Proposition 3.3 and the “unitarity” of  $U$

$$\|p_\lambda(b^*U_x a)\| \leq 4\|\psi_\alpha(b^*b)\| \|\psi_\alpha(a^*a)\| .$$

Consequently

$$\|p_\lambda(b^*U_\mu a)\| \leq 4\|\psi_\alpha(b^*b)\| \|\psi_\alpha(a^*a)\| \|\mu\|_1 .$$

But  $p_\lambda(b^*U_\mu a)$  converges strictly to  $\langle b, U_\mu a \rangle_D$ . The desired inequality follows.  $\square$

**6.11 Proposition.** *Suppose that  $a \in \mathcal{P}_\alpha$  is of the form  $U_g(d)$  for  $d \in \mathcal{P}_\alpha$  and  $g \in C_c(G)$ . Then the function  $x \mapsto U_x a$  is continuous for the norm on  $\mathcal{P}_\alpha$  coming from the  $D_\alpha$ -valued inner product defined by  $\psi_\alpha$ .*

*Proof.* Since  $U_g(d)$  is defined in terms of the norm of  $A$ , a standard calculation shows that  $U_x(U_g d) = U_{L_x g}(d)$  where  $L_x$  is the usual left translation on function on  $G$ . Consequently for any  $b \in \mathcal{P}_\alpha$ ,

$$\begin{aligned} \|\langle b, U_x a \rangle_D - \langle b, a \rangle_D\| &= \|\langle b, U_x a - a \rangle_D\| \\ &= \|\langle b, U_{(L_x g - g)} d \rangle_D\| \leq 4\|\psi(b^*b)\| \|\psi(d^*d)\| \|L_x g - g\|_1 . \end{aligned}$$

But it is a standard fact that  $L$  is strongly continuous on  $L^1(G)$ . From this the above inequality shows that  $U$  is “weakly continuous”. The “strong continuity” then follows in the usual way from the fact that  $U$  is “unitary”. That is,

$$\begin{aligned} \|U_a - a\|_\psi^2 &= \|\langle U_x a - a, U_x a - a \rangle_D\| \\ &= \|\langle a, a \rangle_D - \langle U_x a, a \rangle_D - \langle a, U_x a \rangle_D + \langle a, a \rangle_D\| \\ &= 2\|\langle a - U_x a, a \rangle_D\| \end{aligned}$$

$\square$

As our equivalence bimodule we should surely take the part of  $\mathcal{P}_\alpha$  on which  $U$  is strongly continuous for the norm from  $\psi_\alpha$ , which the above proposition makes clear is still dense

in  $A$  if  $\alpha$  is proper. But since I have not seen how to overcome the main obstacle, which we will discuss shortly, I will avoid the added notational complexity this would require in view of the lack of an answer to Question 6.8. We will just continue to deal with  $\mathcal{P}_\alpha$  itself. Note also that if  $G$  is discrete the issue of strong continuity does not arise.

We now turn to the action of  $A$ . The left action,  $L$ , of  $P_\alpha$  on itself commutes with the right action of  $D_\alpha$ . Let  $b \in Q_\alpha$ . Since  $Q_\alpha$  is a left ideal in  $A$ , the positive linear functional  $a \mapsto \psi_\alpha(b^*ab)$  is defined on all of  $A$ , and so is continuous (lemma 6.1 of [La]). Thus there is a constant,  $K$ , such that

$$\|\psi_\alpha(b^*a^*ab)\| \leq K\|a\|^2 ,$$

and this remains true when  $a$  and  $b$  are restricted to be in  $\mathcal{P}_\alpha$ . Then this says that

$$\|\langle L_a b, L_a b \rangle_D\| \leq K\|a\|^2 ,$$

so that the  $*$ -homomorphism  $L$  of  $\mathcal{P}_\alpha$  into  $B(\bar{\mathcal{P}}_\alpha)$  is continuous, hence of norm 1. We have thus obtained:

**6.12 Proposition.** *For  $a \in \mathcal{P}_\alpha$  the operator  $L_a$  on  $\mathcal{P}_\alpha$  is a bounded operator for the  $D_\alpha$ -valued inner product, and in fact  $\|L_a\| \leq \|a\|$ . Thus  $L_a$  extends to a continuous  $*$ -homomorphism from the closure in  $A$  of  $\mathcal{P}_\alpha$  into  $B(\bar{\mathcal{P}}_\alpha)$ .*

However, in general I do not see why the representation  $L$  on  $\bar{\mathcal{P}}_\alpha$  need be non-degenerate, i.e. why  $L_{\mathcal{P}_\alpha}(\bar{\mathcal{P}}_\alpha)$  need be dense in  $\bar{\mathcal{P}}_\alpha$  for the norm from  $\psi_\alpha$ , although again this can be shown to be true for many examples. This question is closely related to the question mentioned in the comments following Proposition 6.2.

On the other hand,  $U$  and  $L$  do satisfy the usual covariance relation. For  $a, b \in \mathcal{P}_\alpha$  and  $x \in G$  we have

$$U_x(L_a b) = \Delta(x)^{\frac{1}{2}} \alpha_x(ab) = L_{\alpha_x(a)} U_x b ,$$

so that

$$U_x L_a = L_{\alpha_x(a)} U_x .$$

Thus if  $U$  is strongly continuous (for example if  $G$  is discrete) and if the representation  $L$  of  $A$  is non-degenerate, then by the usual universal property [Pe2] we obtain a non-degenerate  $*$ -representation of the crossed product algebra  $A \times_\alpha G$  on  $\bar{\mathcal{P}}_\alpha$ . We will not repeat here the discussion from [Rf6] which indicates that we should actually obtain a representation of the reduced crossed product algebra, since we have more serious difficulties. Namely, we need to know that the algebra  $E$  of compact operators, which is strongly Morita equivalent to  $D_\alpha$ , is contained in (the image of) the crossed product algebra. For this it suffices to show that whenever  $a, b \in \mathcal{P}_\alpha$  then  $\langle a, b \rangle_E$ , defined above, is in  $A \times_\alpha G$ . Now at least symbolically, for  $c \in \mathcal{P}_\alpha$ ,

$$\langle a, b \rangle_E c = a \langle b, c \rangle_D = \int a \alpha_x(b^*) \alpha_x(c) dx = \int a \alpha_x(b^*) \Delta(x)^{-\frac{1}{2}} U_x(c) dx .$$

So we want the function  $x \mapsto a \alpha_x(b) \Delta(x)^{-\frac{1}{2}}$  to be the kernel-function for an operator which lies in  $A \times_\alpha G$ . In [Rf7] this was achieved simply by assuming that this kernel-function is in  $L^1(G, A)$  for  $a$  and  $b$  in a dense subalgebra, much as discussed in Theorem

5.7 above. But under our present more general hypotheses one can find examples where  $a, b \in \mathcal{P}_\alpha$  but the above kernel-function is not in  $L^1(G, A)$ . This does not mean that such a kernel-function could not still represent an element of  $A \times_\alpha G$ . But Exel [E2] has shown that in general it does not. In fact, for the case in which  $G$  is Abelian he gives necessary and sufficient conditions for this to happen. We refer the reader to the very interesting “relative continuity” condition which he shows must hold, and to his discussion of the difficulty of finding a big subspace of mutually relatively continuous elements.

## 7. Square-Integrable Representations.

In this section we study the special case in which the algebra  $A$  is the algebra  $K$  of compact operators. Then  $M(K) = K''$ , which very much simplifies matters. (For certain considerations a more general setting would involve  $C^*$ -algebras  $A$  such that  $M(A)$  is monotone complete [Pe].) The strict topology on  $M(K)$  coincides with the ultra-strong-\* operator topology (p. 76 of [La]). Every bounded increasing net of self-adjoint elements in  $M(K)$  converges in the strong, so ultra-strong-\* and strict, topologies (lemma 6.1.4 of [KR1]), and so for an action  $\alpha$  of a locally compact group  $G$ , every integrable element is proper. That is,  $\mathcal{P}_\alpha = \mathcal{M}_\alpha$  with the notation of the previous sections.

We will show that proper actions are closely related to square-integrable representations of  $G$ . While this is not surprising, it turns out to provide an attractive viewpoint on square-integrable representations.

Let  $K$  be realized as the algebra,  $K(H)$ , of compact operators on a Hilbert space  $H$ . Every automorphism of  $K$  is given by conjugation by an element of  $\mathcal{U}(H)$ , the group of unitary operators on  $H$ , and this unitary operator is unique up to a scalar multiple of modulus 1. Thus if  $\alpha$  is an action of  $G$  on  $K$ , it is given by a projective representation of  $G$  on  $H$ . For our purposes this can be handled most easily [Rf3] by passing to the corresponding extension group. That is, let

$$G_\alpha = \{(x, u) \in G \times \mathcal{U}(H) : \alpha_x(a) = uau^{-1} \text{ for all } a \in K\}.$$

Let  $T$  denote the group of complex numbers of modulus 1. Then we have a short exact sequence

$$0 \rightarrow T \rightarrow G_\alpha \rightarrow G \rightarrow 0,$$

where the map from  $T$  is given by  $t \mapsto (e_G, tI_H)$ , while the map from  $G_\alpha$  is given by  $(x, u) \mapsto x$ . From the topologies on  $T$  and  $G$  we obtain a locally compact topology on  $G_\alpha$  making the exact sequence of groups a topological exact sequence. The map  $(x, u) \mapsto u$  gives an ordinary unitary representation of  $G_\alpha$  on  $H$ . The corresponding action on  $K$  will be the pull-back to  $G_\alpha$  of the action  $\alpha$  of  $G$ . By passing to  $G_\alpha$  we can in this way always assume that  $\alpha$  comes from an ordinary representation. Because  $T$  is compact, it is easily seen that this passage does not affect whether the action on  $K$  is proper.

Thus from now on we always assume that we have an ordinary unitary representation,  $U$ , of  $G$ , on a Hilbert space  $H = H_U$ , and that  $\alpha$  is the corresponding action on  $K$ . Our immediate goal is to find necessary and sufficient conditions on  $U$  such that  $\alpha$  is proper.

Suppose now that  $a \in \mathcal{P}_\alpha^+$ ,  $a \neq 0$ . Since  $a$  is a compact operator and  $\mathcal{P}_\alpha^+$  is a hereditary cone, it follows that each of the spectral projections of  $a$ , and each of their subprojections, is in  $\mathcal{P}_\alpha^+$ . Thus  $\mathcal{P}_\alpha$  contains enough rank-one projections to generate a  $C^*$ -subalgebra of  $K$  containing  $a$ . Consequently for many purposes we can focus on the rank-one projections in  $\mathcal{P}_\alpha^+$ . Let  $p$  be such, and let  $\xi$  be a unit vector spanning the range of  $p$ . For any  $\eta, \zeta \in H$  we will write  $\langle \eta, \zeta \rangle_K$  for the rank-one operator determined by  $\eta$  and  $\zeta$ . For convenience we take the inner-product on  $H$  to be linear in the second variable. Thus we set

$$\langle \eta, \zeta \rangle_K \zeta_0 = \eta \langle \zeta, \zeta_0 \rangle$$

for  $\zeta_0 \in H$ . Then  $p = \langle \xi, \xi \rangle_K$ . Since  $p \in \mathcal{P}_\alpha$ , there is a constant  $k_\xi$  such that

$$(7.1) \quad \int \lambda(x) \alpha_x(p) dx \leq k_\xi^2 I_H$$

for  $\lambda \in \mathcal{B}$ . Thus for any  $\eta \in H$

$$k_\xi^2 \|\eta\|^2 \geq \int \lambda(x) \langle \alpha_x(p) \eta, \eta \rangle dx = \int \lambda(x) |\langle U_x \xi, \eta \rangle|^2 dx.$$

Because of the definition of  $\mathcal{B}$  it follows that  $x \mapsto \langle U_x \xi, \eta \rangle$  is in  $L^2(G)$ .

**7.2 Notation.** For any  $\xi, \eta \in H$  we define the corresponding coefficient function  $c_{\xi\eta}$  by

$$c_{\xi\eta}(x) = \langle U_x \xi, \eta \rangle.$$

With this notation, and with  $\xi$  now any vector in the range of  $p$ , we see from the above that there is a constant  $k_\xi$  such that

$$(7.3) \quad \|c_{\xi\eta}\|_2 \leq k_\xi \|\eta\|$$

for any  $\eta \in H$ .

Suppose now that  $g \in L^1(G) \cap L^2(G)$ , and let  $U_g$  denote the integrated form of  $U$ . Then it follows that

$$\begin{aligned} |\langle U_g \xi, \eta \rangle| &= \left| \int \bar{g}(x) c_{\xi\eta}(x) dx \right| \\ &\leq \|g\|_2 k_\xi \|\eta\|, \end{aligned}$$

for every  $\eta \in H$ . Consequently,

$$(7.4) \quad \|U_g \xi\| \leq k_\xi \|g\|_2.$$

Suppose, conversely, that  $\xi \in H$  and that we know that an inequality of form 7.4 holds for all  $g \in L^1 \cap L^2(G)$ . Running the above argument backward, we obtain 7.3, and then 7.1, so that  $p \in \mathcal{P}_\alpha$ . We have thus obtained:

**7.5 Proposition.** *Let  $U$  be a unitary representation of  $G$  on  $H$ , with corresponding action  $\alpha$  on  $K = K(H)$ . Let  $\xi \in H$ , and let  $p_\xi = \langle \xi, \xi \rangle_K$ . Then  $p_\xi \in \mathcal{P}_\alpha$  if and only if there is a constant,  $k_\xi$ , such that*

$$\|c_{\xi\eta}\|_2 \leq k_\xi \|\eta\|$$

*for every  $\eta \in H$ , or equivalently, such that*

$$\|U_g \xi\| \leq k_\xi \|g\|_2$$

*for every  $g \in L^1 \cap L^2(G)$ .*

**7.6 Definition.** We will call a vector  $\xi$  satisfying these equivalent conditions (i.e. 7.3 and 7.4) a  *$U$ -bounded* vector. We will denote the set of  $U$ -bounded vectors by  $\mathcal{B}_U$ .

This definition is closely related to the definition of bounded elements in Hilbert algebras [D,Rf1,Pj,Cm]. Compare also with Connes' treatment of square-integrable representations of foliations beginning on page 573 of [Cn]. (For a recent variation see definition 1.3 of [Bi].) It is clear that  $\mathcal{B}_U$  is a linear subspace, possibly not closed, in  $H$ . If  $\xi \in \mathcal{B}_U$  and  $x \in G$ , then for  $\eta \in H$

$$c_{U_x \xi, \eta}(y) = \langle U_y U_x \xi, \eta \rangle = c_{\xi\eta}(yx).$$

Thus

$$(7.7) \quad \|c_{U_x \xi, \eta}\|_2 = \Delta(x)^{-1/2} \|c_{\xi\eta}\|_2.$$

Consequently  $\mathcal{B}_U$  is carried into itself by  $U$ .

**7.8 Definition.** We will say that a unitary representation  $U$  of  $G$  on  $H$  is *square-integrable* if  $\mathcal{B}_U$  is dense in  $H$ .

This is exactly the special case for groups of Combes' definition in 1.7 of [Cm] for left Hilbert algebras.

We will see that this definition is equivalent to the more traditional definitions in those situations where they have been given. But conditions 7.3 and 7.4, which do not seem to have been especially emphasized before, are very convenient. In view of our discussion just before 7.1 of the fact that if  $a \in \mathcal{P}_\alpha^+$  then all its spectral projections must be in  $\mathcal{P}_\alpha^+$ , we almost immediately obtain from Proposition 7.5:

**7.9 Theorem.** *Let  $U$  be a unitary representation of  $G$  on  $H$ , with corresponding action  $\alpha$  on  $K(H)$ . Then  $\alpha$  is proper if and only if  $U$  is square-integrable.*

We now begin to show the relation with the usual definitions of square-integrable representations given in the irreducible or cyclic cases [D, Ro, DM, M, Pj, Ca]. Let  $\xi \in \mathcal{B}_U$ . Define an operator,  $T_\xi$ , from  $L^1 \cap L^2(G)$  into  $H$  by

$$T_\xi(g) = U_g(\xi).$$

The definition of  $\mathcal{B}_U$  says that  $T_\xi$  is bounded for the  $L^2$ -norm, with  $\|T_\xi\| \leq k_\xi$ . Furthermore, if we denote by  $L$  the left regular representation of  $G$  on  $L^2(G)$ , we have

$$T_\xi(L_x g) = U_{L_x g}(\xi) = U_x U_g(\xi) = U_x(T_\xi(g)).$$

Thus  $T_\xi$  extends to a bounded intertwining operator from  $L^2(G)$  to  $H$ , which we still denote by  $T_\xi$ . Since  $U$  is assumed to be non-degenerate, it is clear that the closure of the range of  $T_\xi$  is exactly the cyclic subspace in  $H$  generated by  $\xi$ . If we now form the polar decomposition of  $T_\xi$ , then the partially isometric term will be a unitary intertwining operator from some closed invariant subspace of  $L^2(G)$  onto this cyclic subspace. (See VI.13.14 of [FD].) We thus obtain:

**7.10 Proposition.** *Let  $U$  be a unitary representation of  $G$ , and let  $\xi \in \mathcal{B}_U$ . Then the restriction of  $U$  to the cyclic subspace generated by  $\xi$  is unitarily equivalent to a subrepresentation of the left regular representation of  $G$ .*

To clarify the situation a bit more, we note the following analogue of Proposition 5.1, which follows immediately from 7.4:

**7.11 Proposition.** *Let  $U$  and  $V$  be two unitary representations of  $G$ , and let  $T$  be a bounded intertwining operator from  $U$  to  $V$ . Then*

$$T(\mathcal{B}_U) \subseteq \mathcal{B}_V.$$

**7.12 Corollary.** *Let  $U$  be a unitary representation of  $G$ , and let  $P$  be the projection operator onto a  $U$ -invariant subspace. Then  $P(\mathcal{B}_U) \subseteq \mathcal{B}_U$ .*

The following proposition is almost immediate from Definitions 7.8.

**7.13 Proposition.** *The direct sum of a (possibly infinite) family of square-integrable unitary representations of  $G$  is square-integrable.*

From Corollary 7.12 and the usual process of decomposing a representation into a (possibly infinite) direct sum of cyclic representations, we immediately obtain one direction of:

**7.14 Theorem.** *The square-integrable representations of  $G$  are exactly those which are unitarily equivalent to a (possibly infinite) direct sum of copies of subrepresentations of the left regular representation of  $G$ .*

*Proof.* We must show the converse. The crux is to show that the left regular representation of  $G$  is square-integrable. Let  $\xi \in C_c(G)$ . For any  $g \in L^1 \cap L^2(G)$  we have

$$(L_g \xi)(x) = \int g(xy) \xi(y^{-1}) dy = \int (R_y g)(x) (J \bar{\xi})(y) dy = (R_{J \bar{\xi}} g)(x),$$

where  $(R_y g)(x) = \Delta(y)^{1/2} g(xy)$  so that  $R$  is the unitary right-regular representation, and  $J$  is the Tomita–Takesaki operator [KR] defined by  $(J \eta)(y) = \Delta(y)^{-1/2} \overline{\eta(y^{-1})}$ . Thus

$$\|L_g \xi\|_2 = \|R_{J \bar{\xi}} g\|_2 \leq \|J \xi\|_1 \|g\|_2.$$

(More generally, we see that if  $\xi \in L^2(G)$  and  $\|J \xi\|_1 < \infty$ , then  $\xi \in \mathcal{B}_L$ .) Since  $C_c(G)$  is dense in  $L^2(G)$ , this shows that  $L$  is square-integrable. The appearance of  $J$  in the above calculation indicates that something somewhat interesting is happening. We will pursue this matter shortly.

From Corollary 7.12 it follows that every subrepresentation of  $L$  is square-integrable. The proof is then completed by Proposition 7.13.  $\square$

The most common definitions of square-integrable representations just involves the condition that  $c_{\xi\eta} \in L^2(G)$  for some  $\xi, \eta \in H_U$  (and  $c_{\xi\eta} \neq 0$ ). Our tiny contribution to this aspect is to point out now that by using the basic notion of the graph of an unbounded operator, we can avoid explicit use of the theory of unbounded operators and their polar decomposition when dealing with this condition. A very similar argument, involving an irreducible representation, appears in the appendix of [GMP].

**7.15 Proposition.** *Let  $U$  be a unitary representation of  $G$  on  $H$ . Let  $\xi, \eta \in H$ , and suppose that  $c_{\xi\eta} \in L^2(G)$ . Let  $H_\xi$  denote the cyclic subspace generated by  $\xi$ , and replace  $\eta$  by its projection in this subspace. Then the restriction of  $U$  to the cyclic subspace  $H_\eta$  generated by (the new)  $\eta$  is unitarily equivalent to a subrepresentation of  $L$ .*

*Proof.* We can assume that  $H = H_\xi$ . Note that if  $c_{\xi\zeta} \equiv 0$  for some  $\zeta \in H$  then  $\zeta = 0$ , since  $\xi$  is cyclic. Let

$$\begin{aligned}\mathcal{D} &= \{\zeta \in H : c_{\xi\zeta} \in L^2(G)\}, \\ \Gamma &= \{(\zeta, c_{\xi\zeta}) : \zeta \in \mathcal{D}\}.\end{aligned}$$

Then  $\Gamma$  is a closed subspace of  $H \oplus L^2(G)$ . For suppose  $\{\zeta_n\}$  is a sequence in  $\mathcal{D}$  such that  $\zeta_n$  converges to  $\zeta \in H$  and  $c_{\xi\zeta_n}$  converges to  $h \in L^2(G)$ . Then from the definition of  $c_{\xi\zeta_n}$  we see that  $\{c_{\xi\zeta_n}\}$  converges uniformly pointwise to  $c_{\xi\zeta}$ . Thus  $c_{\xi\zeta} = h$ , so  $c_{\xi\zeta} \in L^2(G)$ .

It is easily checked that  $\Gamma$  is a  $U \oplus L$  invariant subspace of  $H \oplus L^2(G)$ . Consider the operator  $Q$  with domain  $H$  defined by

$$\zeta \mapsto (\zeta, 0) \mapsto (\zeta', c_{\xi\zeta'}) \mapsto c_{\xi\zeta'}$$

where the second arrow is the orthogonal projection from  $H \oplus L^2(G)$  onto  $\Gamma$ . Then  $Q$  is clearly a norm-decreasing operator which intertwines  $U$  and  $L$ . If  $Q\zeta = 0$  then  $\zeta$  is clearly orthogonal to  $\mathcal{D}$ . Thus  $Q$  is injective on the closure,  $\bar{\mathcal{D}}$ , of  $\mathcal{D}$ , which is a  $U$ -invariant subspace. From the polar decomposition of the bounded operator  $Q$  we obtain an isometric intertwining operator from  $\bar{\mathcal{D}}$  to  $L^2(G)$ . Clearly  $H_\eta \subseteq \bar{\mathcal{D}}$ .  $\square$

We remark that if  $U$  is irreducible, then  $Q$  is already a multiple of an isometry from  $\bar{\mathcal{D}}$ . If  $G$  is unimodular, then  $\|c_{\xi\eta}\|_2 = \|c_{\eta\xi}\|_2$  in all  $\xi, \eta \in H$ . From this it follows easily that  $\mathcal{B}_U = H$  in this case (for irreducible  $U$ ).

We also remark that if  $N$  denotes the operator  $\zeta \mapsto \zeta'$  for  $\zeta'$  as above, then  $N$  is a norm-decreasing intertwining operator on  $H$ , and that  $Q\zeta = c_{\xi N\zeta}$ . But

$$c_{\xi N\zeta}(x) = \langle U_x \xi, N\zeta \rangle = \langle U_x N\xi, \zeta \rangle = c_{N\xi, \zeta},$$

so  $N\xi \in \mathcal{B}_U$  in view of the properties of  $Q$ .

If  $U$  is irreducible, then  $N$  must be a multiple of an isometry from  $H$ . In particular, every element of  $H$  would be in the range of  $N$ , so that  $c_{\xi\zeta} \in L^2(G)$  for all  $\zeta \in H$ . When this is combined with part *iii* of the restatement in [BT] of theorem 3 of [DM], this says that if  $\xi$  is “admissible” [BT], then  $\xi \in \mathcal{B}_U$ .

The next proposition ties the situation a bit more closely to the discussion of the previous sections. Its proof is an immediate application of the definitions.

**7.16 Proposition.** *Let  $U$  be a unitary representation of  $G$  on  $H$ , with corresponding action  $\alpha$  on  $K$ , and let  $\xi, \eta \in H$ . Then  $\langle \eta, \xi \rangle_K \in \mathcal{Q}_\alpha (= \mathcal{N}_\alpha)$  iff  $\xi \in \mathcal{B}_U$ .*

The left regular representation of  $G$  comes from the action of  $G$  on  $C_\infty(G)$  by left translation, which is proper, together with the invariant unbounded (for  $G$  not compact) trace on  $C_\infty(G)$  consisting of Haar measure. This suggests that perhaps we obtain square-integrable representations from other proper actions and invariant traces. But the occurrence of the operator  $J$  in the proof of Theorem 3.14 should warn us of possible difficulties. On the other hand, because traces are “measure-theoretic” we will see that we can deal with integrable actions – the full force of being proper is not important here.

Let  $\alpha$  be an action of  $G$  on a  $C^*$ -algebra  $A$ . We recall [KR2, Pe2] that a trace on  $A$ , possibly unbounded, is a function  $\tau$  from  $A^+$  to  $[0, \infty]$  with the expected properties. The correct set-up for us here appears to involve the following definition. (See [DM].)

**7.17 Definition.** Let  $\tau$  be a trace on  $A$ . Then  $\tau$  is said to be  $\Delta$ -semi-invariant for the action  $\alpha$  if

$$\tau(\alpha_x(a)) = \Delta(x)^{-1} \tau(a)$$

for all  $a \in A^+$  and  $x \in G$ .

Much as earlier, we set  $\mathcal{M}_\tau^+ = \{a \in A^+ : \tau(a) < \infty\}$ ,  $\mathcal{N}_\tau = \{a \in A : a^*a \in \mathcal{M}_\tau^+\}$ , and  $\mathcal{M}_\tau = \text{span} \mathcal{M}_\tau^+$ . Then  $\mathcal{M}_\tau = \mathcal{N}_\tau^2$ , and  $\mathcal{M}_\tau \cap A^+ = \mathcal{M}_\tau^+$ , and  $\tau$  extends to  $\mathcal{M}_\tau$ . Notice that if  $\tau$  is  $\Delta$ -semi-invariant for  $\alpha$ , then  $\mathcal{M}_\tau^+$  is carried into itself by  $\alpha$ , and similarly for  $\mathcal{N}_\tau$  and  $\mathcal{M}_\tau$ .

We recall (proposition 6.1.3 of [Pe2]) that if  $\tau$  is a lower semi-continuous trace (or weight) on  $A$ , then the GNS construction works to produce a non-degenerate  $*$ -representation of  $A$ . We denote its Hilbert space by  $H_\tau$ , but do not use specific notation for the representation (i.e. we use module notation). Each element  $a \in \mathcal{N}_\tau$  determines an element of  $H_\tau$ , but our notation will not distinguish between elements of  $\mathcal{N}_\tau$  and their images in  $H_\tau$ . The first parts of the following theorem are basically well-known.

**7.18 Theorem.** *Let  $\alpha$  be an action of  $G$  on a  $C^*$ -algebra  $A$ , and let  $\tau$  be a  $\Delta$ -semi-invariant lower semi-continuous trace on  $A$ , with GNS representation on  $H_\tau$ . Define a unitary representation  $U$  of  $G$  on  $H_\tau$  by*

$$U_x(a) = \Delta(x)^{1/2} \alpha_x(a)$$

*for  $a \in \mathcal{N}_\tau$ . Then  $U$  is strongly continuous. Furthermore, every element of  $\mathcal{M}_\alpha \cap \mathcal{M}_\tau$  is  $U$ -bounded. If  $\alpha$  is integrable, then  $U$  is square-integrable.*

*Proof.* We remark that  $\mathcal{N}_\tau$  may be very small, even just  $\{0\}$ . But because  $\tau$  is a trace,  $\mathcal{M}_\tau$  is a two-sided ideal in  $A$ , as is then  $\mathcal{N}_\tau$ . (See 6.2.1 of [Pe2].)

By the semi-continuity of  $\tau$  the image of  $\mathcal{M}_\tau$  in  $\mathcal{H}_\tau$  is dense (see 7.4.1 of [D]), and the representation of  $A$  on  $\mathcal{H}_\tau$  is non-degenerate. For  $a \in \mathcal{M}_\tau$ ,  $b \in \mathcal{M}_\tau^+$ , and  $x \in G$  we have

$$\langle U_x(a), b \rangle_\tau = \tau(\alpha_x(a^*)b) = \tau(b^{1/2} \alpha_x(a^*) b^{1/2})$$

by proposition 5.2.2 of [Pe2]. But  $c \mapsto \tau(b^{1/2} c b^{1/2})$  is a positive linear functional defined on all of  $A$ , so continuous. From this and the fact that  $\mathcal{M}_\tau$  is the span of  $\mathcal{M}_\tau^+$  we see that  $U$  is weakly continuous. Thus  $U$  is strongly continuous since it is unitary.

Suppose now that  $a \in \mathcal{M}_\alpha^+ \cap \mathcal{M}_\tau$  and that  $g \in L^1 \cap L^2(G)$ . Assume further that  $g$  has compact support. Since  $U$  is strongly continuous, the integrated form,  $U_g$ , is defined. Then

$$\begin{aligned} \|U_g a\|^2 &= \langle a, U_{g^{**}g} a \rangle_\tau = \tau(a^{1/2} \alpha_{\Delta^{1/2}(g^{**}g)}(a) a^{1/2}) \\ &\leq \tau(a) \|\alpha_{\Delta^{1/2}(g^{**}g)}(a)\| \leq \tau(a) k_a \|\Delta^{1/2}(g^{**}g)\|_\infty, \end{aligned}$$

since  $a \in \mathcal{M}_\alpha^+$ . But for each  $x \in G$  we have

$$\Delta^{1/2}(x) |(g^{**}g)(x)| \leq \Delta^{1/2}(x) \int |\bar{g}(y)g(yx)| dy \leq \Delta^{1/2}(x) \|g\|_2 \|g(\cdot x)\|_2 = \|g\|_2^2.$$

Thus  $\|\Delta^{1/2}(g^{**}g)\|_\infty \leq \|g\|_2^2$ . Putting this together, we obtain

$$\|U_g a\|^2 \leq \tau(a) k_a \|g\|_2^2.$$

Since any  $g \in L^1 \cap L^2(G)$  can be approximated by ones of compact support, simultaneously in  $L^1$  and  $L^2$  norm, this inequality holds for all  $g \in L^1 \cap L^2(G)$ . This says that  $a$  as vector in  $\mathcal{H}_\tau$  is  $U$ -bounded. Thus we see that every element of  $\mathcal{M}_\alpha^+ \cap \mathcal{M}_\tau$  is  $U$ -bounded. Since  $\mathcal{M}_\tau$  is the span of  $\mathcal{M}_\tau^+$ , and similarly for  $\mathcal{M}_\alpha$ , it follows that every element of  $\mathcal{M}_\alpha \cap \mathcal{M}_\tau$  is  $U$ -bounded.

Suppose now that  $\alpha$  is integrable. To show that  $U$  is square-integrable it suffices to show that  $\mathcal{M}_\alpha \cap \mathcal{M}_\tau$  is dense in  $\mathcal{H}_\tau$ . Since  $\mathcal{M}_\tau$  is an ideal and  $\mathcal{M}_\alpha$  is hereditary, we see that  $\mathcal{M}_\alpha \mathcal{M}_\tau \mathcal{M}_\alpha \subset \mathcal{M}_\alpha \cap \mathcal{M}_\tau$ . We show that in  $\mathcal{H}_\tau$  every element of  $\mathcal{M}_\tau$  can be approximated by elements of  $\mathcal{M}_\alpha \mathcal{M}_\tau \mathcal{M}_\alpha$ . Since, as noted above,  $\mathcal{M}_\tau$  is dense in  $\mathcal{H}_\tau$ , this will conclude the proof.

Let  $I$  denote the norm-closure of  $\mathcal{M}_\tau$  in  $A$ , so that  $I$  is an  $\alpha$ -invariant ideal in  $A$ . Since  $\mathcal{M}_\alpha$  is assumed dense in  $A$ , it follows that  $\mathcal{M}_\alpha I \mathcal{M}_\alpha$  is dense in the  $C^*$ -algebra  $I$ . But  $\mathcal{M}_\alpha I \mathcal{M}_\alpha \subset \mathcal{M}_\alpha$  since  $\mathcal{M}_\alpha$  is hereditary. Thus  $\mathcal{M}_\alpha \cap I$  is dense in  $I$ . Although we don't need it here, we note that we have essentially proven the following analogue of Proposition 5.5:

**7.19 Proposition.** *Let  $\alpha$  be an integrable action of  $G$  on  $A$ , and let  $I$  be an  $\alpha$ -invariant ideal in  $A$ . Then the action  $\alpha$  of  $G$  on  $I$  is integrable.*

We continue with the proof of Theorem 7.18. Pick a positive approximate identity of norm 1 for  $I$ . Since  $\mathcal{M}_\alpha \cap I$  is a dense  $*$ -subalgebra of  $I$ , we can approximate the approximate identity by elements of  $\mathcal{M}_\alpha \cap I$  to obtain a self-adjoint approximate identity of norm 1 consisting of elements of  $\mathcal{M}_\alpha \cap I$ . We can then square this approximate identity to obtain one which is positive. We denote the resulting approximate identity in  $\mathcal{M}_\alpha \cap I$  by  $\{e_\lambda\}$ .

Let  $b \in \mathcal{M}_\tau^+$ . Then  $e_\lambda b e_\lambda \in \mathcal{M}_\alpha \cap \mathcal{M}_\tau$ . We show that  $\{e_\lambda b e_\lambda\}$  converges to  $b$  in  $\mathcal{H}_\tau$ . Now, using heavily that  $\tau$  is tracial, we have

$$\|b - e_\lambda b e_\lambda\|_\tau^2 = \tau(b^2 - 2e_\lambda b e_\lambda + b e_\lambda^2 b e_\lambda^2) = \tau(b^2) - \tau(b^{1/2}(2e_\lambda b e_\lambda - e_\lambda^2 b e_\lambda^2)b^{1/2}).$$

But  $b^{1/2} \in \mathcal{N}_\tau$  and so  $a \mapsto \tau(b^{1/2}ab^{1/2})$  is a positive linear functional defined everywhere on  $A$ , and so continuous. Since  $2e_\lambda be_\lambda - e_\lambda^2 be_\lambda^2$  converges to  $b$  in norm, we see that  $e_\lambda be_\lambda$  does indeed converge to  $b$  in  $\mathcal{H}_\tau$ .  $\square$

It is not at all clear to me how much of the above can be done if  $\tau$  is only a weight instead of a trace. As mentioned before Question 6.9, even the question as to whether the unitary representation  $U$  is strongly continuous seems quite delicate.

Let  $\alpha$  now be the proper action of  $G$  on  $A = C_\infty(G)$  by *right* translation, so  $(\alpha_x(f))(y) = f(yx)$ . Let  $\tau$  be the trace on  $A$  defined by *left* Haar measure. Then  $\pi$  is  $\Delta$ -semi-invariant for  $\alpha$ . Thus we obtain what we already know:

**7.20 Corollary.** *The right regular representation of  $G$  on  $L^2(G)$  is square-integrable.*

But we know that the left-regular representation too is square-integrable. (It is equivalent to the right regular representation.) Even more, the left action of a subgroup of  $G$  on  $L^2(G)$  should be square-integrable. We can relate this to Theorem 7.18 as follows. Let  $\alpha$  be an action of  $G$  on a  $C^*$ -algebra  $A$ , and let  $\tau$  be a trace on  $A$  which we now suppose to be actually  $\alpha$ -invariant. Suppose that  $d$  is an unbounded positive invertible operator affiliated with  $A$  in the sense of Woronowicz [Wo]. (See also Baaĵ [Ba].) For our purposes this means that we have a morphism, say  $\theta$ , from  $D = C_\infty(\mathbb{R})$  to  $A$  (that is, a  $*$ -homomorphism from  $D$  into  $M(A)$  such that  $\theta(D)A$  is dense in  $A$ ) together with a strictly positive  $\delta \in C(\mathbb{R})$  (where  $C(\mathbb{R})$  denotes the algebra of possibly unbounded continuous function on  $\mathbb{R}$ ) acting by pointwise multiplication as an unbounded operator on  $C_\infty(\mathbb{R})$  with domain  $D_0 = C_c(\mathbb{R})$ . Then  $d$  is defined to be the closure of the operator on  $A$  with domain  $\theta(D_0)A$  defined by

$$d(\theta(\varphi)a) = (\theta(\delta\varphi))a$$

for  $\varphi \in D_0$ ,  $a \in A$ . Let  $C$  denote the range of  $\theta$ , and set  $C_0 = \theta(D_0)$ . For our present purposes we require that  $d$  be central, that is, that  $C \subseteq ZM(A)$ , where  $ZM(A)$  denotes the center of  $M(A)$ . Now  $\alpha$  carries  $ZM(A)$  into itself, and  $d$  can be represented by an unbounded continuous function on the maximal ideal space of the center. From this point of view it is clear what we mean by  $\alpha_x(d)$  for  $x \in G$ .

**7.21 Definition.** We say that a central positive operator  $d$  affiliated with  $A$ , via the morphism  $\theta$  from  $C_\infty(\mathbb{R})$  to  $A$ , is  $\Delta$ -semi-invariant for  $\alpha$  if the range of  $\theta$  is carried into itself by  $\alpha_x$  and

$$\alpha_x(d) = \Delta(x)d$$

for all  $x \in G$ .

As one example we have:

**7.22 Proposition.** *Let  $\alpha$  be a proper action of  $G$  on a locally compact space  $M$ , and so on  $A = C_\infty(M)$ . Assume that  $M/\alpha$  is paracompact. Then there exists a central positive invertible operator affiliated with  $A$  which is  $\Delta$ -semi-invariant for  $\alpha$ .*

*Proof.* We imitate the construction of “Bruhat approximate cross-sections”. For each  $\alpha$ -orbit choose an element of  $C_c(M)^+$  which is not everywhere 0 on that orbit. The images in  $M/\alpha$  of the sets where these functions are non-zero form an open cover of  $M/\alpha$ . By

paracompactness there is a locally finite subcover. Let  $b$  denote the sum of the functions for this subcover. So  $b$  is a continuous positive function, with the property that its support meets the preimage in  $M$  of any compact subset of  $M/\alpha$  in a compact set, and it is not everywhere 0 on any orbit. Define a function  $h$  on  $M$  by

$$h(m) = \int_G b(\alpha_y^{-1}(m)) \Delta^{-1}(y) dy.$$

The integrand has compact support for each  $m$ , so  $h$  is well-defined. From the properties of  $b$  it is clear that  $h$  is positive, continuous, and nowhere 0, so invertible. Furthermore, for  $x \in G$  we have

$$(\alpha_x(h))(m) = \int b(\alpha_y^{-1}(\alpha_x^{-1}(m))) \Delta^{-1}(y) dy = \Delta(x)h(m).$$

Thus  $h$  is  $\Delta$ -semi-invariant. (The requirement on domains is easily checked.)  $\square$

It is natural to wonder whether there are interesting extensions of this construction for non-commutative  $A$ 's.

We now continue the general development.

For  $c \in ZM(A)^+$  define  $\tau_c$  on  $A^+$  by

$$\tau_c(a) = \tau(ca) = \tau(a^{1/2}ca^{1/2}).$$

It is clear that  $\tau_c$  is a trace on  $A$ . Note that if  $a \in \mathcal{M}_\tau$  then  $ca \in \mathcal{M}_\tau$  since  $ca = a^{1/2}ca^{1/2} \leq \|c\|a$  and  $\mathcal{M}_\tau$  is hereditary. Thus  $\mathcal{M}_{\tau_c} \supseteq \mathcal{M}_\tau$ . It is easily seen that since  $\tau$  is lower semi-continuous, so is  $\tau_c$ .

Now let  $d$  be a central positive operator affiliated with  $A$ , by means of the morphism  $\theta$  from  $D$  to  $A$  and  $\delta \in C(\mathbb{R})$ . Analogously to what we did in the previous sections let  $\mathcal{B}(C_0) = \{c \in C_0 : 0 \leq c \leq 1\}$ , with its usual upward directed order. Then  $\{\tau_{dc} : c \in \mathcal{B}(C_0)\}$  is an increasing net of lower semi-continuous traces on  $A$ , and so we can define  $\tau_d$  to be their upper bound. Thus  $\tau_d$  is a lower semi-continuous trace on  $A$ . For  $a \in A^+$  and  $c_0 \in C_0^+$  we have

$$\tau_d(c_0a) = \lim_c \tau(dcc_0a) = \lim_c \tau(a^{1/2}dcc_0a^{1/2}) = \tau((dc_0)a)$$

where  $c$  ranges over  $\mathcal{B}(C_0)$ . In particular,  $C_0\mathcal{M}_\tau \subseteq \mathcal{M}_{\tau_d}$ .

We are assuming that  $d$  is invertible. It is then easy to see that  $\tau$  comes from  $\tau_d$  by the above procedure using  $d^{-1}$ . That is,  $\tau = (\tau_d)_{d^{-1}}$ . In particular,  $C_0\mathcal{M}_{\tau_d} \subseteq \mathcal{M}_\tau$ . Now  $C_0 = C_0^3$ , so

$$C_0\mathcal{M}_\tau = C_0^3\mathcal{M}_\tau \subseteq C_0^2\mathcal{M}_{\tau_d} \subseteq C_0\mathcal{M}_\tau.$$

Thus  $C_0\mathcal{M}_\tau = C_0\mathcal{M}_{\tau_d}$ . By a calculation which is computationally simpler than that near the end of the proof of Theorem 7.18 one sees that  $C_0\mathcal{M}_\tau$  is dense in  $\mathcal{H}_\tau$ . One must just notice that the fact that  $C_0$  is in  $M(A)$  rather than  $A$  causes no difficulties. From what we have seen,  $C_0\mathcal{M}_\tau$  is also dense in  $\mathcal{H}_{\tau_d}$ . Let us define an operator,  $T$ , from  $\mathcal{H}_{\tau_d}$  to  $\mathcal{H}_\tau$  by first defining it on  $C_0\mathcal{M}_\tau$  by

$$T(ca) = (d^{1/2}c)a.$$

One checks that this is well-defined as follows. Given  $\sum c_i a_i = \sum c'_j a'_j$ , choose  $c \in C_0$  such that  $cc_i = c_i$  and  $cc'_j = c'_j$  for all  $i, j$ . Then

$$T(\sum c_i a_i) = (d^{1/2}c) \sum c_i a_i = T(\sum c'_j a'_j).$$

Then

$$\langle Tc_1 a_1, Tc_2 a_2 \rangle_\tau = \tau((dc_2^* c_1) a_2^* a_1) = \langle c_1 a_1, c_2 a_2 \rangle_{\tau_d}.$$

Thus on its domain  $T$  is isometric. But its domain and range are dense in  $\mathcal{H}_{\tau_d}$  and  $\mathcal{H}_\tau$  respectively. So  $T$  extends to a unitary operator between them.

Now suppose that  $\alpha$  is an action of  $G$  on  $A$ , that  $\tau$  is  $\alpha$ -invariant, and that  $d$  is  $\Delta$ -semi-invariant for  $\alpha$ . In particular,  $\alpha$  carries  $C_0$  into itself and is an automorphism of the directed set  $\mathcal{B}(C_0)$ . Then for  $c \in \mathcal{B}(C_0)$  and  $a \in A^+$  we have

$$\begin{aligned} \tau_{dc}(\alpha_x(a)) &= \tau(dc\alpha_x(a)) = \tau(\alpha_x((\alpha_{x^{-1}}(dc))a)) \\ &= \tau(\alpha_{x^{-1}}(dc)a) = \Delta(x)^{-1} \tau(d\alpha_{x^{-1}}(c)a) = \Delta(x)^{-1} \tau_{d\alpha_{x^{-1}}(c)}(a). \end{aligned}$$

On taking the limit over  $\mathcal{B}(C_0)$  we obtain

$$\tau_d(\alpha_x(a)) = \Delta(x^{-1}) \tau_d(a),$$

that is,  $\tau_d$  is  $\Delta$ -semi-invariant for  $\alpha$ .

We can now apply Theorem 7.18 to conclude that the unitary representation  $V$  on  $\mathcal{H}_{\tau_d}$  coming from  $\alpha$  is square-integrable if  $\alpha$  is integrable. Of course  $V$  is defined by

$$V_x(a) = \Delta(x)^{1/2} \alpha_x(a).$$

At the same time we have the unitary representation  $U$  on  $\mathcal{H}_\tau$  defined by

$$U_x(a) = \alpha_x(a).$$

But consider the unitary operator  $T$  defined several paragraphs ago. For  $c \in C_0$  and  $a \in \mathcal{M}_\tau$  we have

$$\begin{aligned} U_x(T(ca)) &= U_x((d^{1/2}c)a) = \alpha_x((d^{1/2}c)a) = \alpha_x(d^{1/2}c) \alpha_x(a) \\ &= \Delta(x)^{1/2} d^{1/2} \alpha_x(c) \alpha_x(a) = T(\Delta(x)^{1/2} \alpha_x(ca)) = T(V_x(ca)). \end{aligned}$$

Thus  $T$  is an intertwining operator, and the two representations are equivalent. We have thus demonstrated:

**7.23 Theorem.** *Let  $\alpha$  be an action of  $G$  on a  $C^*$ -algebra  $A$ , and let  $\tau$  be an  $\alpha$ -invariant lower semi-continuous trace on  $A$ . Let  $U$  be the corresponding unitary representation of  $G$  on  $\mathcal{H}_\tau$ . If  $\alpha$  is integrable, and if there is a central positive invertible operator affiliated to  $A$  which is  $\Delta$ -semi-invariant, then  $U$  is square-integrable.*

We will see a reflection of this theorem in the next section. Upon applying Proposition 7.22 we obtain:

**7.24 Corollary.** *Let  $\alpha$  be a proper action of  $G$  on a locally compact space  $M$  such that  $M/\alpha$  is paracompact. For every positive  $\alpha$ -invariant Radon measure  $\mu$  on  $M$  the corresponding unitary representation of  $G$  on  $L^2(M, \mu)$  is square-integrable.*

As one (unsurprising) application of the earlier Theorem 7.18 we can consider the canonical trace on the algebra of compact operators, whose GNS Hilbert space is the space of Hilbert–Schmidt operators.

**7.25 Corollary.** *Let  $G$  be a unimodular group and let  $U$  be a square-integrable representation of  $G$  on  $H$ . Let  $\alpha$  be the corresponding action on  $K(H)$ , which is integrable. Then the corresponding unitary representation on the space of Hilbert–Schmidt operators is square-integrable.*

## 8. The Orthogonality Relations.

In this section we examine what the orthogonality relations for square-integrable representations look like from the vantage point of the previous section.

Let  $U$  be a representation of  $G$  on  $H$ , and let  $\alpha$  be the corresponding action on  $K = K(H)$ . Then  $M(K)$  consists of the bounded operators on  $H$ , and  $M(K)^\alpha$  is exactly the algebra of intertwining operators for  $U$ . Let  $\xi$  and  $\eta \in \mathcal{B}_U$ . From Proposition 7.5 and polarization it follows rapidly that  $\langle \xi, \eta \rangle_K \in \mathcal{M}_\alpha$ , and so the integral

$$(8.1) \quad \int \alpha_x(\langle \xi, \eta \rangle_K) dx$$

converges in the strong operator topology to an operator in  $M(K)^\alpha$ . From the vantage point of this paper, the orthogonality relations are concerned with identifying to some extent this intertwining operator. The reason is that for any  $\zeta, \omega \in H$  we have

$$(8.2) \quad \begin{aligned} \left\langle \int \alpha_x(\langle \xi, \eta \rangle_K) dx \zeta, \omega \right\rangle &= \int \overline{\langle U_x \eta, \zeta \rangle} \langle U_x \xi, \omega \rangle dx \\ &= \langle c_{\eta\zeta}, c_{\xi\omega} \rangle_{L^2(G)}, \end{aligned}$$

so that any answer will say something about the inner-product of the coefficient functions. It is quite clear that if  $\xi$  and  $\eta$  come from two subrepresentations which are disjoint (have no non-zero intertwining operators) then 8.1 must be 0 and so  $c_{\xi\omega}$  and  $c_{\eta\zeta}$  must be orthogonal. Since any two representations can be viewed as subrepresentations of their direct sum, we obtain:

**8.3 Proposition.** *(The “first orthogonality relation”.) Let  $U$  and  $V$  be representations of  $G$  on  $H_U$  and  $H_V$ . Suppose that  $U$  and  $V$  are disjoint. Then for any  $\xi, \omega \in H_U$  with  $\xi$   $U$ -bounded, and for any  $\eta, \zeta \in H_V$  with  $\eta$   $V$ -bounded, the coefficient functions  $c_{\xi\omega}$  and  $c_{\eta\zeta}$  are orthogonal in  $L^2(G)$ .*

So we concentrate on the “second orthogonality relation”. We introduced after Theorem 7.9 the bounded operators  $T_\xi$  for  $\xi \in \mathcal{B}_U$ . Let us calculate  $T_\xi^*$ . For  $\eta \in H$  and  $g \in L^1 \cap L^2(G)$  we have

$$\langle g, T_\xi^* \eta \rangle = \langle L_g \xi, \eta \rangle = \langle g, c_{\xi\eta} \rangle.$$

Thus

$$T_\xi^* \eta = c_{\xi\eta}.$$

For  $\xi, \eta \in \mathcal{B}_U$  and  $\zeta, \omega \in H$  it follows that

$$\langle T_\xi T_\eta^* \zeta, \omega \rangle = \langle c_{\eta\zeta}, c_{\xi\omega} \rangle.$$

From 8.2 we then obtain:

**8.4 Proposition.** *Let  $U$  be a unitary representation of  $G$ . For  $\xi, \eta \in \mathcal{B}_U$  we have*

$$\int \alpha_x(\langle \xi, \eta \rangle_K) dx = T_\xi T_\eta^*.$$

We now examine what this says for the left regular representation,  $L$ , of  $G$ . Let  $h \in \mathcal{B}_L$ . From the calculation in the proof of Theorem 7.14, but now with  $g \in C_c(G)$ , we see that, at the level of functions,

$$T_h g = R_{J\bar{h}} g.$$

In interpreting this, note that  $J$  is an isometry, so that  $J\bar{h} \in L^2(G)$ . The operator  $R_{J\bar{h}}$ , which is initially defined only on, say,  $C_c(G)$ , is bounded, by our assumption on  $h$ , and so extends to a bounded operator on all of  $L^2(G)$ . With this understanding, we have

$$T_h = R_{J\bar{h}}.$$

The second orthogonality relation for the left regular representation can then be considered to be the following statement:

**8.5 Theorem.** *For  $f, g \in \mathcal{B}_L$  we have*

$$\int \alpha_x(\langle f, g \rangle_K) dx = R_{J\bar{f}} R_{J\bar{g}}^*.$$

We remark that this is closely related to the result in example 2.1 of [Rf7] (where the  $\rho$  there is not unitary).

We now relate this to Exel's example of section 13 of [E2]. Suppose that  $G$  is Abelian. We conjugate  $L$  by the Fourier transform so that it acts on  $L^2(\hat{G})$ , by pointwise multiplication of characters. Upon applying the Plancherel theorem, we see that equation 7.4 becomes

$$\|\hat{g}\xi\|_2 \leq k_\xi \|\hat{g}\|_2$$

for all  $g \in L^1 \cap L^2(G)$  and  $\xi \in L^2(\hat{G})$ . This will hold exactly if  $\xi \in L^\infty(\hat{G})$ . Thus under this picture  $\mathcal{B}_L = L^\infty \cap L^2(\hat{G})$ . From the fact that  $G$  is Abelian it is easily seen that  $R_{J\bar{f}} = L_f$ , so that the relation in Theorem 8.5 reads

$$\int \alpha_x(\langle f, g \rangle_K) dx = L_{f \star g^*}.$$

But on  $L^2(\hat{G})$  the operator  $L_{f \star g^*}$  is just pointwise multiplication by the Fourier transform of  $f \star g^*$ . If we change notation so that now  $f, g \in L^\infty \cap L^2(\hat{G}) = \mathcal{B}_L$ , and if we let  $M_{f\bar{g}}$  denote the operator of pointwise multiplication by  $f\bar{g}$ , we obtain

$$\int \alpha_x(\langle f, g \rangle_K) dx = M_{f\bar{g}}.$$

Let  $f, g \in L^\infty \cap L^2(\hat{G})$ , and choose any  $h \in L^\infty \cap L^2(\hat{G})$  such that  $\|h\|_2 = 1$ . Then

$$\langle f, g \rangle_K = \langle f, h \rangle_K \langle h, g \rangle_K,$$

and so from Proposition 7.5 and the fact that  $\mathcal{P}_\alpha$  is an algebra we see that the “big fixed-point-algebra” will contain  $M_{f\bar{g}}$ . In particular, if  $\hat{G}$  is compact, the big fixed-point algebra is exactly  $L^\infty(\hat{G})$ . This is Exel’s example. Certainly  $L^\infty(\hat{G})$  is too big. As Exel indicates, it is well-known that  $K(L^2(G)) \times_\alpha G$  is isomorphic to  $K(L^2(G)) \otimes C_\infty(\hat{G})$ , which has  $\hat{G}$  as primitive ideal space. Since the primitive ideal spaces of strongly Morita equivalent algebras are homeomorphic [Rf3], it is impossible for  $L^\infty(\hat{G})$  to be strongly Morita equivalent to an ideal in  $K \times_\alpha G$  (unless  $G$  is finite). We refer the reader to [E2] for substantial further exploration of this situation.

We wish to consider next the case in which  $U$  is irreducible. But we first make some observations about the general case which will be useful for that purpose. For any  $\xi, \eta \in \mathcal{B}_U$ , the operator  $T_\xi^* T_\eta$  on  $L^2(G)$  intertwines  $L$ . For any  $g \in L^1 \cap L^2(G)$  we have

$$T_\xi^* T_\eta g = T_\xi^* U_g \eta = L_g T_\xi^* \eta = g * c_{\xi\eta}.$$

Note that this implies that  $c_{\xi\eta}$  is in the closure of the range of  $T_\xi^* T_\eta$ . Note next that  $c_{\xi\eta}(x^{-1}) = \bar{c}_{\eta\xi}(x)$ , so that  $c_{\xi\eta}$  is in  $L^2(G, \Delta^{-1}dx)$  as well as in  $L^2(G)$ , where by  $\Delta^{-1}dx$  we denote right Haar measure. Now for any  $\varphi \in L^2(G)$  and  $\psi \in L^2(G, \Delta^{-1}dx)$  we have

$$|\varphi * \psi(x)| = \left| \int \varphi(y) \psi(y^{-1}x) dy \right| \leq \|\varphi\|_2 \|\psi(\cdot^{-1})\|_2.$$

This says that convolution is well defined and jointly continuous from  $L^2(G) \times L^2(G, \Delta^{-1}dx)$  to  $L^\infty(G)$ . But if  $\varphi, \psi \in C_c(G)$ , then  $\varphi * \psi \in C_c(G) \subseteq C_\infty(G)$ . Since  $C_c(G)$  is dense in  $L^2(G)$ , it follows that  $\varphi * \psi \in C_\infty(G)$  for all  $\varphi \in L^2(G)$  and  $\psi \in L^2(G, \Delta^{-1}dx)$ .

In particular, for  $\xi, \eta \in \mathcal{B}_U$  and for any  $\varphi \in L^2(G)$ , the function  $\varphi * c_{\xi\eta}$  is continuous. But we saw above that  $g * c_{\xi\eta} = T_\xi^* T_\eta g$  for  $g \in L^1 \cap L^2(G)$ . Let  $\{g_n\}$  be a sequence in  $L^1 \cap L^2(G)$  which converges to  $\varphi$ . As seen above,  $g_n * c_{\xi\eta}$  converges uniformly to  $\varphi * c_{\xi\eta}$ , but it also converges in  $L^2$ -norm to  $T_\xi^* T_\eta \varphi$ . (The main concern here is the fact that  $c_{\xi\xi}$  need not be in  $L^1(G)$ .) We thus obtain:

**8.6 Proposition.** *Let  $U$  be a unitary representation of  $G$ . For any  $\xi, \eta \in \mathcal{B}_U$  and any  $\varphi \in L^2(G)$ ,*

$$T_\xi^* T_\eta \varphi = \varphi * c_{\xi\eta},$$

and  $\varphi * c_{\xi\eta} \in C_\infty(G) \cap L^2(G)$ . Thus the range of  $T_\xi^* T_\eta$  consists entirely of functions in  $C_\infty(G)$ .

We now consider the case in which  $U$  is an irreducible representation of  $G$ . In this case, since  $\mathcal{B}_U$  is an invariant subspace, it must be dense in  $H$  as soon as it contains one non-zero vector, which we will assume. Let  $\xi, \eta \in \mathcal{B}_U$ . Since  $T_\xi T_\eta^*$  is an intertwining operator, it must be a scalar multiple of the identity operator. We denote the scalar by  $\gamma(\xi, \eta)$ , so that

$$T_\xi T_\eta^* = \gamma(\xi, \eta) I_H.$$

We wish to obtain a more revealing expression for  $\gamma$ . We follow the general outline of the treatment given in [Ca], but our details are more elementary because of our use of  $\mathcal{B}_U$ .

By suitably normalizing  $\xi$ , we can arrange that  $T_\xi^*$  is an isometry. Then  $T_\xi^* T_\xi$  is a projection operator on  $L^2(G)$ , intertwining  $L$ . The restriction of  $L$  to the range of  $T_\xi^* T_\xi$  is a subrepresentation of  $L$  which is unitarily equivalent to  $U$ . As seen above, the range of  $T_\xi^* T_\xi$  consists entirely of continuous functions. But now, since  $T_\xi^*$  is an isometry, this range is a closed subspace of  $L^2(G)$ . But  $T_\xi^* T_\xi$  is given by right convolution by  $c_{\xi\xi}$ . We have thus obtained:

**8.7 Proposition.** *Let  $U$  be a square-integrable irreducible representation of  $G$ . For any  $\xi \in \mathcal{B}_U$  normalized so that  $T_\xi^*$  is an isometry, right convolution by  $c_{\xi\xi}$  is a projection of  $L^2(G)$  onto a closed subspace consisting entirely of continuous functions, on which  $L$  is unitarily equivalent to  $U$ .*

Let  $H_\xi$  denote the range of the isometry  $T_\xi^*$ . (Note that  $c_{\xi\xi} = T_\xi^* \xi$  so that  $c_{\xi\xi} \in H_\xi$ .) For every  $\varphi \in H_\xi$  and every  $x \in G$  we see above that

$$\varphi(x) = \varphi * c_{\xi\xi}(x) = \int \bar{c}_{\xi\xi}(x^{-1}y) \varphi(y) dy = \langle L_x c_{\xi\xi}, \varphi \rangle.$$

The second equality says exactly that  $H_\xi$  is a “reproducing-kernel Hilbert space” on  $G$ , with reproducing kernel  $c_{\xi\xi}$ . The third equality says that the map  $x \mapsto L_x c_{\xi\xi}$  is a “coherent state” for  $H_\xi$ . (See [Al] for a recent review of coherent states, with many interesting examples.)

Suppose now that  $\zeta, \omega \in \mathcal{B}_U$ . Then from 8.2 but with the roles of the vectors interchanged, and from the definition of  $\gamma(\zeta, \omega)$ , we obtain

$$\begin{aligned} \gamma(\omega, \zeta) \langle \xi, \xi \rangle &= \langle c_{\omega\xi}, c_{\xi\xi} \rangle = \int \bar{c}_{\omega\xi}(y) c_{\xi\xi}(y) dy = \int c_{\xi\omega}(y^{-1}) \bar{c}_{\xi\xi}(y^{-1}) dy \\ &= \int \bar{c}_{\xi\xi}(y) c_{\xi\omega}(y) \Delta(y^{-1}) dy = \langle \Delta^{-1/2} c_{\xi\xi}, \Delta^{-1/2} c_{\xi\omega} \rangle = \langle \Delta^{-1/2} T_\xi^* \zeta, \Delta^{-1/2} T_\xi^* \omega \rangle. \end{aligned}$$

Thus we have

$$\gamma(\omega, \zeta) = \|\xi\|^{-2} \langle \Delta^{-1/2} T_\xi^* \zeta, \Delta^{-1/2} T_\xi^* \omega \rangle.$$

Notice that this is all well-defined, since as seen above,

$$T_\xi^* \zeta = c_{\xi\zeta} \in L^2(G) \cap L^2(G, \Delta^{-1} dx)$$

so that  $\Delta^{-1/2} c_{\xi\zeta} \in L^2(G)$ . In conclusion, we obtain:

**8.8 Theorem.** *Let  $U$  be a square-integrable irreducible representation of  $G$ . Let  $\xi \in \mathcal{B}_U$ , normalized so that  $T_\xi^*$  is an isometry. Then for  $\eta, \zeta \in \mathcal{B}_U$  we have*

$$\int \alpha_x(\langle \eta, \zeta \rangle_K) dx = \|\xi\|^{-2} \langle \Delta^{-1/2} T_\xi^* \zeta, \Delta^{-1/2} T_\xi^* \eta \rangle I_H.$$

We remark that if we choose  $\eta$  and  $\zeta$  so that

$$\langle \Delta^{-1/2} T_\xi^* \zeta, \Delta^{-1/2} T_\xi^* \eta \rangle = \|\xi\|^2,$$

then we obtain

$$\int \alpha_x(\langle \eta, \zeta \rangle_K) dx = I_H.$$

This is just another way of writing the familiar “resolution of the identity” from the theory of coherent states.

If  $G$  is unimodular, we see that the right-hand-side of the equation of Theorem 8.8 simplifies to

$$\|\xi\|^{-2} \langle \zeta, \eta \rangle I_H,$$

and now  $\|\xi\|^2$  is the familiar formal dimension of  $U$ . (See [D,Rf1].)

If  $G$  is not unimodular the right-hand-side of the equation of Theorem 8.8 is a bit unattractive because the vectors and inner-product of the right-hand side are taken in  $L^2(G)$ , not  $H$ . But the considerations just before the statement of Theorem 8.8 show that if  $\eta \in \mathcal{B}_U$  then  $\Delta^{-1} T_\xi^* \eta \in L^2(G)$ . Thus we can define an unbounded operator,  $K$ , with dense domain  $\mathcal{B}_U$  by

$$K^{-1} \eta = \|\xi\|^{-2} T_\xi \Delta^{-1} T_\xi^* \eta.$$

Then one can check that  $K$  is a positive operator, and the right-hand side of the equation of Theorem 8.8 can be rewritten as

$$\langle \zeta, K \eta \rangle I_H = \langle K^{-1/2} \zeta, K^{-1/2} \eta \rangle I_H.$$

This is the form given in theorem 3 of [DM], or theorem 4.3 of [Ca], or theorem 2 of [BT]. We omit the details about domains and the fact that  $K$  is independent of the choice of  $\xi$ . But one can check that, as expected from [DM],  $K$  is  $\Delta^{-1}$ -semi-invariant, reflecting the situation for the left regular representation seen earlier.

We remark that in [Mo] Moore has given orthogonality relations for factor square-integrable representations. But his orthogonality relations are not for the coefficient functions as defined here. So it is not clear to me how his results relate to those given here.

We conclude this section by showing that the possible difficulty mentioned before Definition 1.2, namely that it may happen that  $a \in \mathcal{M}$  but  $|a| \notin \mathcal{M}$ , actually occurs even in the present setting of square-integrable representations.

Let  $G$  be the “ $ax + b$ ” group. So  $G$  is  $\mathbb{R} \times \mathbb{R}^+$  with product given by

$$(p, s)(q, t) = (p + sq, st).$$

The modular function of  $G$  is  $\Delta(p, s) = s^{-1}$ . It is well-known that  $G$  has two inequivalent square-integrable irreducible representations. We consider one of them. It has many

models. For our purposes the most elementary approach to what we need seems to be given by theorem 2 of [BT], so we use the model used there. The Hilbert space is  $H = L^2(\mathbb{R}^+, dt/t)$ , and the representation is given by

$$(\pi_{(p,s)}\xi)(t) = e(pt)\xi(st),$$

where by definition  $e(t) = \exp(2\pi it)$ . Let  $K$  denote the unbounded operator on  $H$  defined by

$$(K\xi)(t) = t\xi(t).$$

One can check that  $K$  is  $\Delta^{-1}$ -invariant.

Applying theorem 2 of [BT] and Theorem 8.8 above to this particular situation, we find that, up to multiplication by a positive scalar,

$$\int_G \alpha_x(T) dx = \text{Tr}(K^{-1}T)I_H$$

for any positive compact operator  $T$ , where  $\alpha$  is the action of conjugation by  $\pi$ . As in our earlier discussion of the irreducible case, this gives essentially an ordinary weight. We can now study this weight independently of the fact that it comes from a group representation.

To simplify our analysis, we make the change of variables  $r = e^{-t}$ . Then our Hilbert space becomes  $L^2(\mathbb{R})$  for Lebesgue measure, and  $D = K^{-1}$  is the operator of pointwise multiplication by  $t \mapsto e^t$ . We denote our weight by  $\psi$ . It is now given by

$$\psi(T) = \text{Tr}(DT).$$

Since  $D$  is unbounded, we must make precise what this means. We can do this conveniently in terms of spectral projections of  $D$ . For our purposes the following works well. For each integer  $n \geq 1$  let  $E_n$  denote the orthogonal projection onto the subspace of  $L^2(\mathbb{R})$  consisting of the functions supported in  $[-n, -n+1] \cup [n-1, n]$ . Thus the  $E_n$ 's are mutually orthogonal and sum to  $I$ . For each  $n$  the operator  $DE_n$  is a bounded positive operator. For any positive bounded operator  $T$  we take  $\psi(T)$  to mean

$$\psi(T) = \sum \text{Tr}(DE_n T).$$

In particular,  $\psi$  is lower semi-continuous. We let  $\mathcal{M}^+ = \{T \in B(H)^+ : \psi(T) < \infty\}$ , and we let  $\mathcal{M}$  be the linear span of  $\mathcal{M}^+$ . For our discussion of square-integrable representations we are most interested in the restriction of  $\psi$  to the algebra  $K(H)$  of compact operators.

**8.9 Theorem.** *With notation as just above, there are  $S, T \in K(H)$  such that  $S, T \in \mathcal{M}^+$  but  $|S - T| \notin \mathcal{M}^+$ .*

*Proof.* For each integer  $n \geq 1$  set  $\xi_n = \chi_{[n-1, n]}$ , and  $\eta_n = \chi_{[-n, -n+1]}$ , where  $\chi$  denotes “characteristic function”. Thus  $\xi_n$  and  $\eta_n$  are unit vectors in the range of  $E_n$ . The following steps are motivated by the example following theorem 2.4 of [Pe1]. Choose a sequence  $\{a_n\}$  of numbers with  $0 \leq a_n \leq 1$  such that  $\sum a_n < \infty$  but  $\sum a_n^{1/2} = \infty$ . For instance,  $a_n = n^{-2}$ . Let  $P_n$  and  $Q_n$  denote the rank-1 projections which are 0 on the

orthogonal complement of  $\{\xi_n, \eta_n\}$ , whereas with respect to the basis  $\{\xi_n, \eta_n\}$  they have matrices

$$\begin{pmatrix} a_n & (a_n - a_n^2)^{1/2} \\ (a_n - a_n^2)^{1/2} & 1 - a_n \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. Then  $(P_n - Q_n)^2$  has matrix

$$\begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix},$$

so that  $|P_n - Q_n|$  has matrix

$$\begin{pmatrix} a_n^{1/2} & 0 \\ 0 & a_n^{1/2} \end{pmatrix}.$$

For any integer  $k$  with  $|k| \geq 1$  set  $d_k = \langle D\xi_k, \xi_k \rangle$  if  $k \geq 1$  and  $d_k = \langle D\eta_{-k}, \eta_{-k} \rangle$  if  $k \leq -1$ . Thus, disregarding  $k = 0$ , we see that  $\{d_n\}$  goes to 0 rapidly as  $k \rightarrow -\infty$ , and to  $+\infty$  rapidly as  $k \rightarrow +\infty$ . We can use the basis  $\{\xi_n, \eta_n\}$  to evaluate  $\psi$  on  $P_n$  and  $Q_n$ , and a quick calculation shows that

$$\psi(P_n) = a_n d_n + (1 - a_n) d_{-n},$$

$$\psi(Q_n) = d_{-n},$$

$$\psi(|P_n - Q_n|) = a_n^{1/2} (d_n + d_{-n}).$$

Set  $S = \sum d_n^{-1} P_n$  and  $T = \sum d_n^{-1} Q_n$ , where the sums are for  $n \geq 1$ . Since  $\{d_n\}$  grows rapidly for positive  $n$ , the sums converge in norm, and  $S, T \in K(H)^+$ . From the above calculations and the properties which we required of  $\{a_n\}$  we see that

$$\psi(S) = \sum a_n + \sum d_n^{-1} (1 - a_n) d_{-n} < \infty,$$

$$\psi(T) = \sum d_n^{-1} d_{-n} < \infty.$$

Thus  $S, T \in \mathcal{M}^+$ . However

$$\psi(|S - T|) = \sum a_n^{1/2} + \sum a_n^{1/2} d_n^{-1} d_{-n} = \infty.$$

Thus  $|S - T| \notin \mathcal{M}^+$ . □

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